Asymptotics in Dynamics, Geometry and PDEs; Generalized Borel Summation
vol. I

edited by
O. Costin, F. Fauvet,
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Introduction

In the last three decades or so, important questions in distinct areas of mathematics such as the local analytic dynamics, the study of analytic partial differential equations, the classification of geometric structures (*e.g.* moduli for holomorphic foliations), or the semi-classical analysis of Schrödinger equation have necessitated in a crucial way to handle delicate asymptotics, with the formal series involved being generally divergent, displaying specific growth patterns for their coefficients, namely of Gevrey type. The modern study of Gevrey asymptotics, for questions originating in geometry or analysis, goes together with an investigation of rich underlying algebraic concepts, revealed by the application of Borel resummation techniques.

Specifically, the study of the Stokes phenomenon has had spectacular recent applications in questions of integrability, in dynamics and PDEs. Some generalized form of Borel summation has been developed to handle the relevant structured expansions – named transseries – which mix series, exponentials and logarithms; these formal objects are in fact ubiquitous in special function theory since the 19th century.

Perturbative Quantum Field Theory is also a domain where recent advances have been obtained, for series and transseries which are of a totally different origin from the ones met in local dynamics and yet display the same sort of phenomena with, strikingly, the very same underlying algebraic objects.

Hopf algebras, *e.g.* with occurrences of shuffle and quasishuffle products that are important themes in the algebraic combinatorics community, appear now natural and useful in local dynamics as well as in pQFT. One common thread in many of the important advances for these questions is the concept of resurgence, which has triggered substantial progress in various areas in the near past.

An international conference took place on October 12th – October 16th, 2009, at the Centro di Ricerca Matematica Ennio De Giorgi, in Pisa, to highlight recent achievements along these ideas.

Here is a complete list of the lectures delivered during this event:

Carl Bender, *Complex dynamical systems*

Filippo Bracci, *One resonant biholomorphisms and applications to quasi-parabolic germs*

David Broadhurst, *Multiple zeta values in quantum field theory*

Jean Ecalle, *Four recent advances in resummation and resurgence theory*

Adam Epstein, *Limits of quadratic rational maps with degenerate parabolic fixed points of multiplier* $\mathrm{e}^{2\pi \mathrm{i}q} \to 1$

Gérard Iooss, *On the existence of quasipattern solutions of the Swift-Hohenberg equation and of the Rayleigh-Benard convection problem*

Shingo Kamimoto, *On a Schrödinger operator with a merging pair of a simple pole and a simple turning point, I: WKB theoretic transformation to the canonical form*

Tatsuya Koike, *On a Schrödinger operator with a merging pair of a simple pole and a simple turning point, II: Computation of Voros coefficients and its consequence*

Dirk Kreimer, *An analysis of Dyson Schwinger equations using Hopf algebras*

Joel Lebowitz, *Time asymptotic behavior of Schrödinger equation of model atomic systems with periodic forcings: to ionize or not*

Carlos Matheus, *Multilinear estimates for the 2D and 3D Zakharov-Rubenchik systems*

Emmanuel Paul, *Moduli space of foliations and curves defined by a generic function*

Jasmin Raissy *Torus actions in the normalization problem*

Javier Ribon, *Multi-summability of unfoldings of tangent to the identity diffeomorphisms*

Reinhard Schäfke, *An analytic proof of parametric resurgence for some second order linear equations*

Mitsuhiro Shishikura, *Invariant sets for irrationally indifferent fixed points of holomorphic functions*

Harris J. Silverstone, *Kramers-Langer-modified radial JWKB equations and Borel summability*

Yoshitsugu Takei, *On the turning point problem for instanton-type solutions of (higher order) Painlevé equations*

Saleh Tanveer, *Borel summability methods applied to PDE initial value problems*

Jean-Yves Thibon, *Noncommutative symmetric functions and combinatorial Hopf algebras*

The present volume, together with a second one to appear in the same collection, contains five contributions of invited speakers at this conference, reflecting some of the leading themes outlined above.

We express our deep gratitude to the staff of the Scuola Normale Superiore di Pisa and of the CRM Ennio de Giorgi, in particular to the Director of the CRM, Professor Mariano Giaquinta, for their dedicated support in the preparation of this meeting; all participants could thus benefit of the wonderful and stimulating atmosphere in these institutions and around Piazza dei Cavalieri. We are also very grateful for the possibility to publish these two volumes in the CRM series. We acknowledge with thankfulness the support of the CRM, of the ANR project "Resonances", of Université Paris 11 and of the Gruppo di Ricerca Europeo Franco-Italiano: Fisica e Matematica, with also many thanks to Professor Jean-Pierre Ramis. All our recognition for the members of the Scientific Board for the conference: Professors Louis Boutet de Monvel (Univ. Paris 6), Dominique Cerveau (Univ. Rennes), Takahiro Kawai (RIMS, Kyoto) and Stefano Marmi (SNS Pisa).

Pisa, January 2011

Ovidiu Costin, Frédéric Fauvet,
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Complex elliptic pendulum

Carl M. Bender, Daniel W. Hook and Karta Singh Kooner

Abstract. This paper briefly summarizes previous work on complex classical mechanics and its relation to quantum mechanics. It then introduces a previously unstudied area of research involving the complex particle trajectories associated with elliptic potentials.

1 Introduction

In generalizing the real number system to the complex number system one loses the ordering property. (The inequality $z_1 < z_2$ is meaningless if $z_1, z_2 \in \mathbb{C}$.) However, extending real analysis into the complex domain is extremely useful because it makes easily accessible many of the subtle features and concepts of real mathematics. For example, complex analysis can be used to derive the fundamental theorem of algebra in just a few lines. (The proof of this theorem using real analysis alone is long and difficult.) Complex analysis explains why the Taylor series for the function $f(x) = 1/\left(1+x^2\right)$ converges in the finite domain $-1 < x < 1$ even though $f(x)$ is smooth for all real x.

This paper describes and summarizes our ongoing research program to extend conventional classical mechanics to complex classical mechanics. In our exploration of the nature of complex classical mechanics we have examined a large class of analytic potentials and have discovered some remarkable phenomena, namely, that these systems can exhibit behavior that one would normally expect to be displayed only by quantum-mechanical systems. In particular, in our numerical studies we have found that complex classical systems can exhibit tunneling-like behavior.

Among the complex potentials we have studied are periodic potentials, and we have discovered the surprising result that such classical potentials can have band structure. Our work on periodic potentials naturally leads us to study the complex classical mechanics of *doubly*-periodic potentials. We have chosen to examine elliptic potentials because such functions are analytic and thus have well-defined complex continuations. This is a rich and previously unexplored area of mathematical physics, and in the current paper we report some new discoveries.

This paper is organized as follows: In Section 2 we give a brief review of complex classical mechanics focusing on the complex behavior of a classical particle in a periodic potential. In Section 3 we describe the motion of a particle in an elliptic-function potential. Finally, in Section 4 we make some concluding remarks and describe the future objectives of our research program.

ACKNOWLEDGEMENTS. CMB is grateful to Imperial College for its hospitality and to the U.S. Department of Energy for financial support. DWH thanks Symplectic Ltd. for financial support. Mathematica was used to generate the figures in this paper.

2 Previous results on complex classical mechanics

During the past decade there has been an active research program to extend quantum mechanics into the complex domain. Specifically, it has been shown that the requirement that a Hamiltonian be Dirac Hermitian (we say that a Hamiltonian is *Dirac-Hermitian* if $H = H^\dagger$, where $\dagger$ represents the combined operations of complex conjugation and matrix transposition) may be broadened to include complex non-Dirac-Hermitian Hamiltonians that are $\mathcal{PT}$ symmetric. This much wider class of Hamiltonians is physically acceptable because these Hamiltonians possess two crucial features: (i) their eigenvalues are all real, and (ii) they describe unitary time evolution. We say that a Hamiltonian is $\mathcal{PT}$ symmetric if it is invariant under combined spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$ [1].

An example of a class of Hamiltonians that is not Dirac Hermitian but which is $\mathcal{PT}$ symmetric is given by

$$H = p^2 + x^2(ix)^\epsilon \qquad (\epsilon > 0). \tag{2.1}$$

When $\epsilon = 0$, H, which represents the familiar quantum harmonic oscillator, is Dirac Hermitian. While H is no longer Dirac Hermitian when ϵ increases from 0, H continues to be $\mathcal{PT}$ symmetric, and its eigenvalues continue to be real, positive, and discrete [2–11]. Because a $\mathcal{PT}$-symmetric quantum system in the complex domain retains the fundamental properties required of a physical quantum theory, much theoretical research on such systems has been published and recent experimental observations have confirmed some theoretical predictions [12–15].

Complex quantum mechanics has proved to be so interesting that the research activity on $\mathcal{PT}$ quantum mechanics has motivated studies of complex classical mechanics. In the study of complex systems the complex as well as the real solutions to Hamilton's differential equations of motion are considered. In this generalization of conventional classical mechanics, classical particles are not constrained to move along the real axis and may travel through the complex plane.

Early work on the particle trajectories in complex classical mechanics is reported in [4, 16]. Subsequently, detailed studies of the complex extensions of various one-dimensional conventional classical-mechanical systems were undertaken: The remarkable properties of complex classical trajectories are examined in [17–21]. Higher dimensional complex classical-mechanical systems, such as the Lotka-Volterra equations for population dynamics and the Euler equations for rigid body rotation are discussed in [22]. The complex $\mathcal{PT}$-symmetric Korteweg-de Vries equation has also been studied [23–29].

The objective in extending classical mechanics into the complex domain is to enhance our understanding of subtle mathematical and physical phenomena. For example, it was found that some of the complicated properties of chaotic systems become more transparent when extended into the complex domain [30]. Also, studies of exceptional points of complex systems have revealed interesting and potentially observable effects [31,32]. Finally, recent work on the complex extension of quantum probability density constitutes an advance in our understanding of the quantum correspondence principle [33].

An elementary example that illustrates the extension of a conventional classical-mechanical system into the complex plane is given by the classical harmonic oscillator, whose Hamiltonian is given in (2.1) with $\epsilon = 0$. The standard classical equations of motion for this system are

$$\dot{x} = 2p, \quad \dot{p} = -2x. \tag{2.2}$$

However, we now treat the coordinate variable $z(t)$ and the momentum variable $p(t)$ to be *complex* functions of time t. That is, we consider this system to have one *complex* degree of freedom. Thus, the equations of motion become

$$\dot{r} = 2u, \quad \dot{s} = 2v, \quad \dot{u} = -2r, \quad \dot{v} = -2s, \tag{2.3}$$

where the complex coordinate is $x = r + is$ and the complex momentum is $p = u + iv$. For a particle having real energy E and initial position

$r(0) = a > \sqrt{E}$, $s(0) = 0$, the solution to (2.3) is

$$r(t) = a\cos(2t), \quad s(t) = \sqrt{a^2 - E}\,\sin(2t). \tag{2.4}$$

Thus, the possible classical trajectories are a family of ellipses parametrized by the initial position a:

$$\frac{r^2}{a^2} + \frac{s^2}{a^2 - E} = 1. \tag{2.5}$$

Five of these trajectories are shown in Figure 2.1. Each trajectory has the same period $T = \pi$. The degenerate ellipse, whose foci are the turning points at $x = \pm\sqrt{E}$, is the familiar real solution. Note that classical particles may visit the real axis in the classically forbidden regions $|x| > \sqrt{E}$, but that the elliptical trajectories are *orthogonal rather than parallel to the real-x axis.*

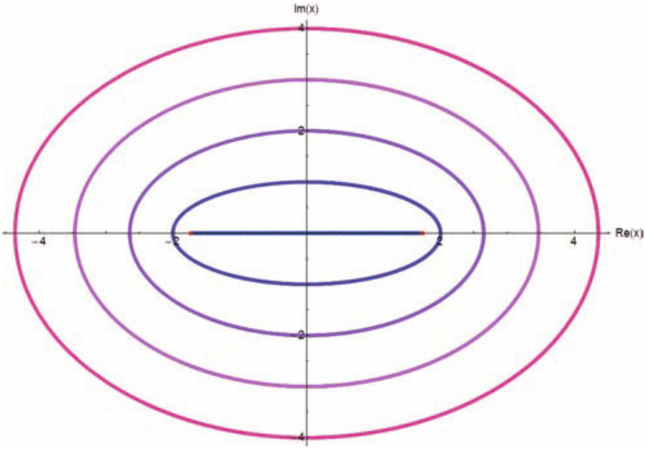

Figure 2.1. Classical trajectories in the complex plane for the harmonic-oscillator Hamiltonian $H = p^2 + x^2$. These trajectories are nested ellipses. Observe that when the harmonic oscillator is extended into the complex-x domain, the classical particles may pass through the classically forbidden regions on the real axis outside the turning points. When the trajectories cross the real axis, they are orthogonal to it.

In general, when classical mechanics is extended into the complex domain, classical particles are allowed to enter the classically forbidden region. However, in the forbidden region there is no particle flow parallel to the real axis and the flow of classical particles is *orthogonal* to the axis. This feature is analogous to the vanishing flux of energy in the case of total internal reflection.

In the case of total internal reflection when the angle of incidence is less than a critical value, there is a reflected wave but no transmitted wave. The electromagnetic field does cross the boundary and this field is attenuated exponentially in a few wavelengths beyond the interface. Although the field does not vanish in the classically forbidden region, there is no flux of energy; that is, the Poynting vector vanishes in the classically forbidden region beyond the interface. We emphasize that in the physical world the cutoff at the boundary between the classically allowed and the classically forbidden regions is not perfectly sharp. For example, in classical optics it is known that below the surface of an imperfect conductor, the electromagnetic fields do not vanish abruptly. Rather, they decay exponentially as functions of the penetration depth. This effect is known as *skin depth* [34].

Another model that illustrates the properties of the classically allowed and classically forbidden regions is the anharmonic oscillator, whose Hamiltonian is

$$H = \tfrac{1}{2}p^2 + x^4. \tag{2.6}$$

For this Hamiltonian there are four turning points, two on the real axis and two on the imaginary axis. When the energy is real and positive, all the classical trajectories are closed and periodic except for two special trajectories that begin at the turning points on the imaginary axis. Four trajectories for the case $E = 1$ are shown in Figure 2.2.

The topology of the classical trajectories changes dramatically if the classical energy is allowed to be complex: When $\operatorname{Im} E \neq 0$, the classical paths are no longer closed. This feature is illustrated in Figure 2.3, which shows the path of a particle of energy $E = 1 + 0.1i$ in an anharmonic potential.

The observation that classical orbits are closed and periodic when the energy is real and open and nonperiodic when the energy is complex was made in [22] and studied in detail in [35]. In these references it is emphasized that the Bohr-Sommerfeld quantization condition

$$\oint_C dx\, p = \left(n + \tfrac{1}{2}\right)\pi \tag{2.7}$$

can only be applied if the classical orbits are closed. Thus, there is a deep connection between real classical energies and the existence of associated real quantum eigenvalues.

It was further argued in [35] that the measurement of a quantum energy is inherently imprecise because of the time-energy uncertainty principle $\Delta E\, \Delta t \gtrsim \hbar/2$. Specifically, since there is not an infinite amount of time

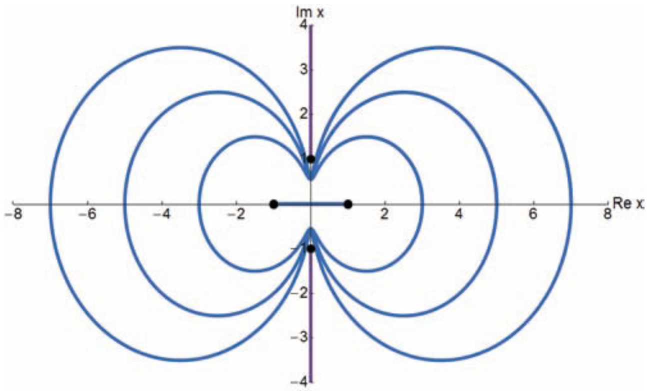

Figure 2.2. Classical trajectories $x(t)$ in the complex-x plane for the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. All trajectories represent a particle of energy $E = 1$. There is one real trajectory that oscillates between the turning points at $x = \pm 1$ and an infinite family of nested complex trajectories that enclose the real turning points but lie inside the imaginary turning points at $\pm i$. (The turning points are indicated by dots.) Two other trajectories begin at the imaginary turning points and drift off to infinity along the imaginary-x axis. Apart from the trajectories beginning at $\pm i$, all trajectories are closed and periodic. All orbits in this figure have the same period $\sqrt{\pi/2}\,\Gamma\left(\frac{1}{4}\right)/\Gamma\left(\frac{3}{4}\right) = 3.70815\ldots$

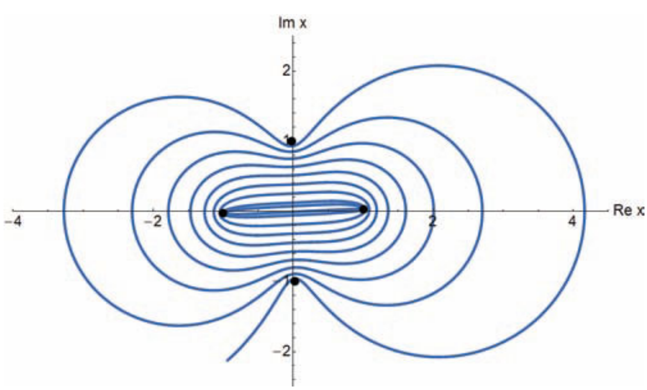

Figure 2.3. A single classical trajectory in the complex-x plane for a particle governed by the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. This trajectory begins at $x = 1$ and represents the complex path of a particle whose energy $E = 1 + 0.1i$ is complex. The trajectory is not closed or periodic. The four turning points are indicated by dots. The trajectory does not cross itself.

in which to make a quantum energy measurement, we expect that the uncertainty in the energy ΔE is nonzero. If we then suppose that this uncertainty has an imaginary component, it follows that in the corresponding classical theory, while the particle trajectories are almost periodic, the orbits do not close exactly. The fact that the classical orbits with complex energy are not closed means that in complex classical mechanics one can observe tunneling-like phenomena that one normally expects to find only in quantum systems.

We illustrate such tunneling-like phenomena by considering a particle in a quartic double-well potential $V(x) = x^4 - 5x^2$. Figure 2.4 shows eight possible complex classical trajectories for a particle of *real* energy $E = -1$. Each of these trajectories is closed and periodic. Observe that for this energy the trajectories are localized either in the left well or the right well and that no trajectory crosses from one side to the other side of the imaginary axis.

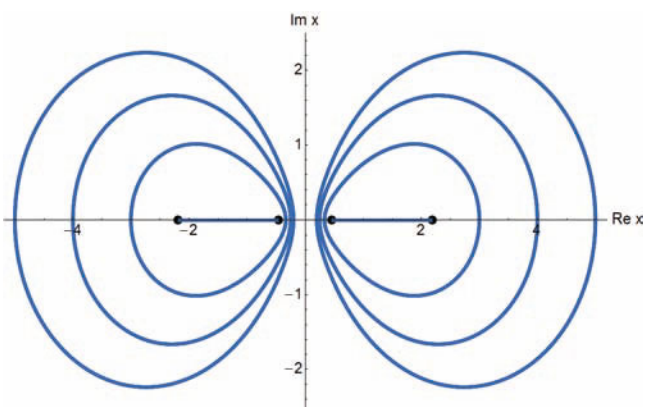

Figure 2.4. Eight classical trajectories in the complex-x plane representing a particle of energy $E = -1$ in the potential $x^4 - 5x^2$. The turning points are located at $x = \pm 2.19$ and $x = \pm 0.46$ and are indicated by dots. Because the energy is real, the trajectories are all closed. The classical particle stays in either the right-half or the left-half plane and cannot cross the imaginary axis. Thus, when the energy is real, there is no effect analogous to tunneling.

What happens if we allow the classical energy to be complex [36]? In this case the classical trajectory is no longer closed. However, it does not spiral out to infinity like the trajectory shown in Figure 2.3. Rather, the trajectory in Figure 2.5 unwinds around a pair of turning points for a characteristic length of time and then crosses the imaginary axis. At this point the trajectory does something remarkable: Rather than continuing

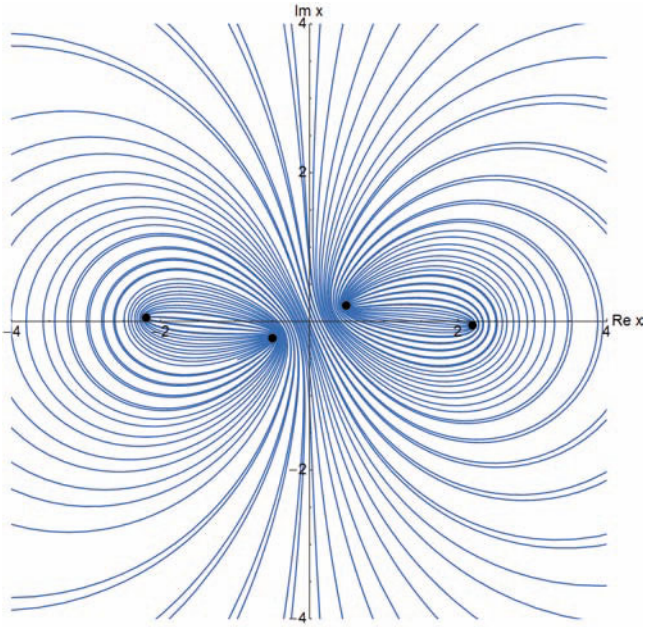

Figure 2.5. Classical trajectory of a particle moving in the complex-x plane under the influence of a double-well $x^4 - 5x^2$ potential. The particle has complex energy $E = -1 - i$ and thus its trajectory does not close. The trajectory spirals outward around one pair of turning points, crosses the imaginary axis, and then spirals inward around the other pair of turning points. It then spirals outward again, crosses the imaginary axis, and goes back to the original pair of turning points. The particle repeats this behavior endlessly but at no point does the trajectory cross itself. This classical-particle motion is analogous to the behavior of a quantum particle that repeatedly tunnels between two classically allowed regions. Here, the particle does not disappear into the classically forbidden region during the tunneling process; rather, it moves along a well-defined path in the complex-x plane from one well to the other.

its outward journey, it spirals *inward* towards the other pair of turning points. Then, never crossing itself, the trajectory turns outward again, and after the same characteristic length of time, returns to the vicinity of the first pair of turning points. This oscillatory behavior, which shares the qualitative characteristics of strange attractors, continues forever but the trajectory never crosses itself. As in the case of quantum tunneling, the particle spends a long time in proximity to a given pair of turning points before crossing the imaginary axis to the other pair of turning points. On average, the classical particle spends equal amounts of time on either side of the imaginary axis. Interestingly, we find that as the imaginary part of the classical energy increases, the characteristic “tunneling” time decreases in inverse proportion, just as one would expect of a quantum particle.

Having described the tunneling-like behavior of a classical particle having complex energy in a double well, we examine the case of such a particle in a periodic potential. Physically, this corresponds to a classical particle in a crystal lattice. A simple physical system that has a periodic potential consists of a simple pendulum in a uniform gravitational field [37]. Consider a pendulum consisting of a bob of mass m and a string of length L in a uniform gravitational field of magnitude g (see Figure 2.6). The gravitational potential energy of the system is defined to be zero at the height of the pivot point of the string. The pendulum bob swings through an angle θ. Therefore, the horizontal and vertical cartesian coordinates X and Y are $X = L\sin\theta$ and $Y = -L\cos\theta$, which gives velocities $\dot{X} = L\dot{\theta}\sin\theta$ and $\dot{Y} = -L\dot{\theta}\cos\theta$. The potential and kinetic energies are $V = -mgL\cos\theta$ and $T = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) = \frac{1}{2}mL^2\dot{\theta}^2$. The Hamiltonian $H = T + V$ for the pendulum is therefore

$$H = \tfrac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta. \tag{2.8}$$

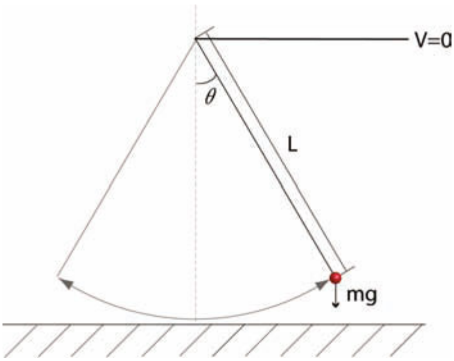

Figure 2.6. Configuration of a simple pendulum of mass m in a uniform gravitational field of strength g. The length of the string is L. The pendulum swings through an angle θ. We define the potential energy to be 0 at the height of the pivot.

Without loss of generality we set $m = 1$, $g = 1$, and $L = 1$ and then make the change of variable $\theta \to x$ to get

$$H = \tfrac{1}{2}p^2 - \cos x, \tag{2.9}$$

where $p = \dot{x}$. The classical equations of motion for this Hamiltonian are

$$\dot{x} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -\sin x. \tag{2.10}$$

The Hamiltonian H for this system is a constant of the motion and thus the energy E is a time-independent quantity.

If we take the energy E to be real, we find that the classical trajectories are confined to cells of horizontal width 2π, as shown in Figure 2.7. This is the periodic analog of Figures 2.1 and 2.2.

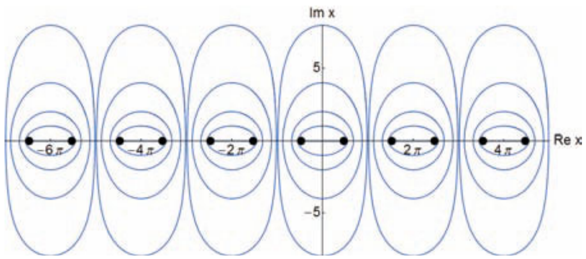

Figure 2.7. Classical trajectories in the complex-x plane for a particle of energy $E = -0.09754$ in a $-\cos x$ potential. The motion is periodic and the particle remains confined to a cell of width 2π. Five trajectories are shown for each cell. The trajectories shown here are the periodic analogs of the trajectories shown in Figures 2.1 and 2.2.

If the energy of the classical particle in a periodic potential is taken to be complex, the particle begins to hop from well to well in analogy to the behavior of the particle in Figure 2.5. This hopping behavior is displayed in Figure 2.8.

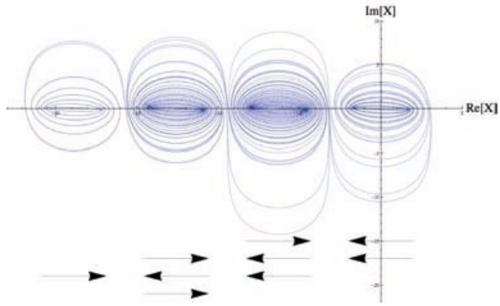

Figure 2.8. A tunneling trajectory for the Hamiltonian (2.9) with $E = 0.1 - 0.15i$. The classical particle hops from well to well in a random-walk fashion. The particle starts at the origin and then hops left, right, left, left, right, left, left, right, right. This is the sort of behavior normally associated with a particle in a crystal at an energy that is not in a conduction band. At the end of this simulation the particle is situated to the left of its initial position. The trajectory never crosses itself.

The most interesting analogy between quantum mechanics and complex classical mechanics is established by showing that there exist narrow conduction bands in the periodic potential for which the quantum particle exhibits resonant tunneling and the complex classical particle exhibits unidirectional hopping [35, 38]. This qualitative behavior is illustrated in Figure 2.9.

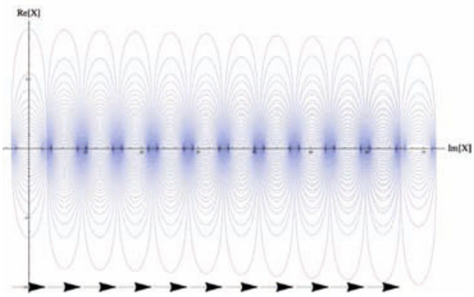

Figure 2.9. A classical particle exhibiting a behavior analogous to that of a quantum particle in a conduction band that is undergoing resonant tunneling. Unlike the particle in Figure 2.8, this classical particle tunnels in one direction only and drifts at a constant average velocity through the potential.

A detailed numerical analysis shows that the classical conduction bands have a narrow but finite width (see Figure 2.10). Two magnified portions of the conduction bands in Figure 2.10 are shown in Figure 2.11. These

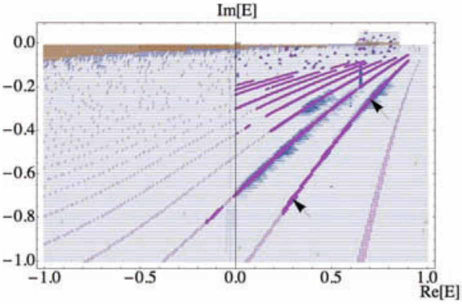

Figure 2.10. Complex-energy plane showing those energies that lead to tunneling (hopping) behavior and those energies that give rise to conduction. Hopping behavior is indicated by a hyphen - and conduction is indicated by an X. The symbol & indicates that no tunneling takes place; tunneling does not occur for energies whose imaginary part is close to 0. In some regions of the energy plane we have done very intensive studies and the X's and -'s are densely packed. This picture suggests the features of band theory: If the imaginary part of the energy is taken to be -0.9, then as the real part of the energy increases from -1 to $+1$, five narrow conduction bands are encountered. These bands are located near $\mathrm{Re}\,E = -0.95,\ -0.7,\ -0.25,\ 0.15,\ 0.7$. This picture is symmetric about $\mathrm{Im}\,E = 0$ and the bands get thicker as $|\mathrm{Im}\,E|$ increases. A total of 68689 points were classified to make this plot. In most places the resolution (distance between points) is 0.01, but in several regions the distance between points is shortened to 0.001. The regions indicated by arrows are blown up in Figure 2.11.

magnifications show that the edges of the conduction bands are sharply defined.

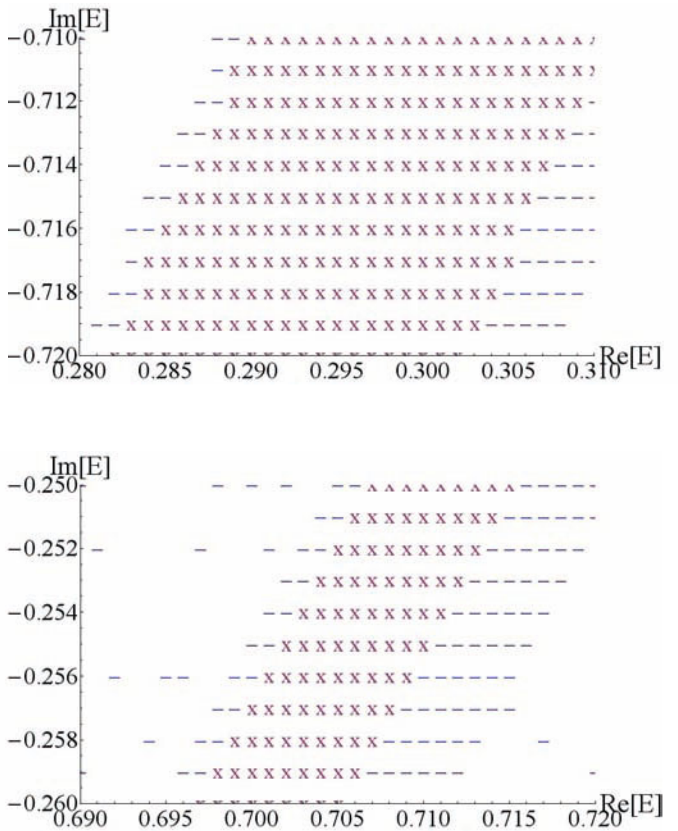

Figure 2.11. Detailed portions of the complex-energy plane shown in Figure 2.10 containing a conduction band. Note that the edge of the conduction band, where tunneling (hopping) behavior changes over to conducting behavior, is very sharp.

3 Classical particle in a complex elliptic potential

Having reviewed in Section 2 the behavior of complex classical trajectories for trigonometric potentials, in this section we give a brief glimpse of the rich and interesting behavior of classical particles moving in elliptic potentials. Elliptic potentials are natural doubly-periodic generalizations of trigonometric potentials.

The Hamiltonian that we have chosen to study is a simple extension of that in (2.9):

$$H = \tfrac{1}{2}p^2 - \mathrm{Cn}(x, k), \tag{3.1}$$

where $\mathrm{Cn}(x, k)$ is a *cnoidal* function [39,40]. When the parameter $k = 0$, the cnoidal function reduces to the singly periodic function $\cos x$ and when $k = 1$, the cnoidal function becomes $\tanh x$. When $0 < k < 1$, the cnoidal function is periodic in both the real and imaginary directions and it is meromorphic (analytic in the finite-x plane except for pole singularities) and has infinitely many simple poles. The real part of the cnoidal potential $\mathrm{Cn}(x, k)$ is plotted in Figure 3.1.

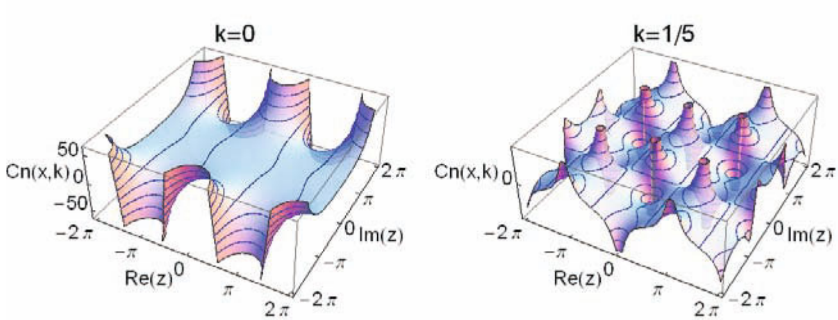

Figure 3.1. Real part of the cnoidal elliptic function $\mathrm{Cn}(x, k)$ in the complex-x plane for two values of k, $k = 0$ and $k = 1/5$. When $k = 0$, this cnoidal function reduces to the trigonometric function $\cos x$, and this function grows exponentially in the imaginary-x direction. When $k > 0$, the cnoidal function is doubly periodic; that is, periodic in the real-x and in the imaginary-x directions. While $\cos x$ is entire, the cnoidal functions for $k \neq 0$ are meromorphic and have periodic simple poles.

The classical particle trajectories satisfy Hamilton's equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\mathrm{Sn}(x, k)\mathrm{Dn}(x, k). \end{aligned} \tag{3.2}$$

The trajectories for $k > 0$ are remarkable in that the classical particles seem to prefer to move vertically rather than horizontally. In Figure 3.2 a trajectory, similar to that in Figure 2.8, is shown for the case $k = 0$. This trajectory is superimposed on a plot of the real part of the cosine potential. The particle oscillates horizontally. In Figure 3.3 a complex trajectory for the case $k = 1/5$ is shown. This more exotic path escapes from the initial pair of turning points, and rather than "tunneling" to a horizontally adjacent pair of turning points, it travels downward. The ensuing wavy vertical motion passes close to many poles before the particle gets captured by another pair of turning points. The particle winds inwards and outwards around these turning points and eventually returns to the original pair of turning points. After escaping from these turning points again, the particle now moves in the positive-imaginary direction.

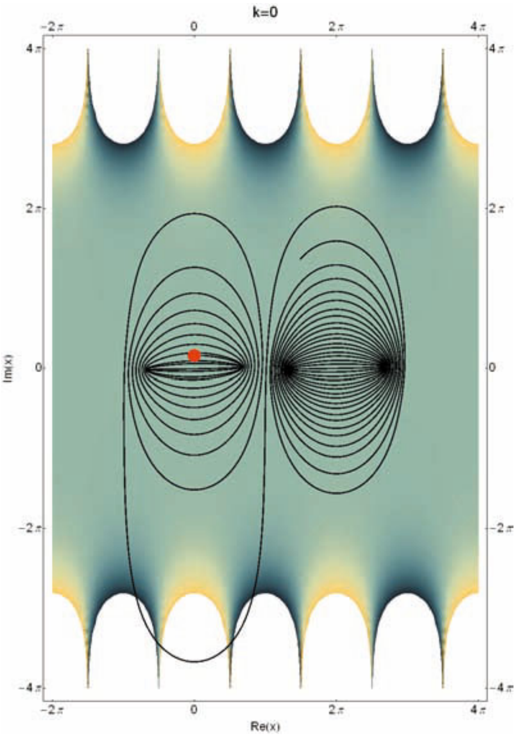

Figure 3.2. Complex classical trajectory for a particle of energy $E = 0.5+0.05i$ in a cosine potential (a cnoidal potential with $k = 0$). The trajectory begins at the red dot at $x = 0.5i$ on the left side of the figure and spirals outward around the left pair of turning points. The trajectory then "tunnels" to the right and spirals inward and then outward around the right pair of turning points. In the background is a plot of the real part of the cosine potential, which is shown in detail in Figure 3.1.

It is clear that classical trajectories associated with doubly periodic potentials have an immensely interesting structure and should be investigated in much greater detail to determine if there is a behavior analogous to band structure shown in Figures 2.10 and 2.11.

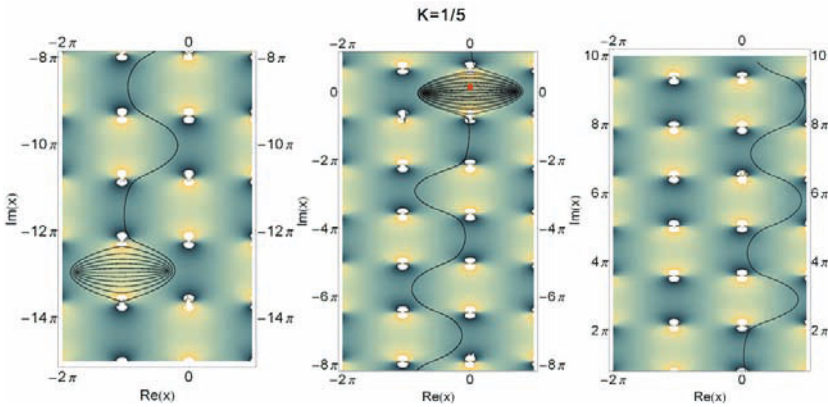

Figure 3.3. Complex classical trajectory for a particle of energy $E = 0.5+0.05i$ in a cnoidal potential with $k = 1/5$. The trajectory begins at the red dot at $x = 0.5i$ near the top of the central pane. The trajectory spirals outward around the two turning points and, when it gets very close to a pole, it suddenly begins to travel downward in a wavy fashion in the negative-imaginary direction. The trajectory bobs and weaves past many simple poles and continues on in the left pane of the figure. It eventually gets very close to a simple pole, gets trapped, and spirals inward towards a pair of turning points. After spiraling inwards, it then spirals outwards (never crossing itself) and goes upward along a path extremely close to the downward wavy path. It is then recaptured by the original pair of turning points in the central pane. After spiraling inwards and outwards once more it now escapes and travels *upward*, bobbing and weaving along a wavy path in the right pane of the figure. Evidently, the particle trajectory strongly prefers to move vertically upward and downward, and not horizontally. The vertical motion distinguishes cnoidal trajectories from those in Figures 2.8, 2.9, and 3.2 associated with the cosine potential.

4 Summary and discussion

The relationship between quantum mechanics and classical mechanics is subtle. Quantum mechanics is essentially wavelike; probability amplitudes are described by a wave equation and physical observations involve such wavelike phenomena as interference patterns and nodes. In contrast, classical mechanics describes the motion of particles and exhibits none of these wavelike features. Nevertheless, there is a deep connection between quantum mechanics and complex classical mechanics. In the complex domain the classical trajectories exhibit a remarkable behavior that is analogous to quantum tunneling.

Periodic potentials exhibit a surprising and intricate feature that closely resembles quantum band structure. It is especially noteworthy that the classical bands, just like the quantum bands, have finite width.

Our early work on singly periodic potentials strongly suggests that further detailed analysis should be done on doubly periodic potentials. Doubly periodic potentials are particularly interesting because they have singularities. Two important and so far unanswered questions are as follows: (i) Does a complex classical particle in a doubly periodic potential undergo a random walk in two dimensions and eventually visit all lattice sites? (ii) Are there special bands of energy for which the classical particle no longer undergoes random hopping behavior and begins to drift in one direction through the lattice?

References

[1] The time evolution operator of such $\mathcal{PT}$ quantum systems has the usual form e^{-iHt} and this operator is unitary with respect to the Hermitian adjoint appropriate for the specific Hamiltonian H. Instead of the Dirac adjoint $\dagger$, the adjoint for a $\mathcal{PT}$-symmetric Hamiltonian H is given by $\mathcal{CPT}$, where $\mathcal{C}$ is a linear operator satisfying the three simultaneous equations: $\mathcal{C}^2 = 1$, $[\mathcal{C}, \mathcal{PT}] = 0$, and $[\mathcal{C}, H] = 0$. The $\mathcal{CPT}$ norm is strictly real and positive and thus the theory is associated with a conventional Hilbert space. The time evolution is unitary because it preserves the $\mathcal{CPT}$ norms of vectors. A detailed discussion of these features of $\mathcal{PT}$ quantum mechanics is presented in [6].

[2] The eigenfunctions $\psi(x)$ of the Hamiltonian in (2.1) obey the differential equation $-\psi''(x) + x^2(ix)^\epsilon\psi(x) = E\psi(x)$. These eigenfunctions are localized and decay exponentially in pairs of Stokes' wedges in the complex-x plane, as is explained in [3]. The eigenfunctions live in L^2 but with the Dirac adjoint $\dagger$ replaced by the $\mathcal{CPT}$ adjoint. The eigenvalues are real and positive (see [7]). There is extremely strong numerical evidence that the set of eigenfunctions form a complete basis, but to our knowledge this result has not yet been rigorously established. Note that for the case $\epsilon = 2$ the potential becomes $-x^4$, but in the complex plane this potential is *not* unbounded below! (The term *unbounded below* cannot be used in this context because the complex numbers are not ordered.) $\mathcal{PT}$ quantum mechanics has many qualitative features, such as arbitrarily fast time evolution, that distinguish it from conventional Dirac-Hermitian quantum mechanics. These features are discussed in [6].

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Parabolic attitude

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Abstract. Being parabolic in complex dynamics is not a state of fact, but it is more an attitude. In these notes we explain the philosophy under this assertion.

1 Introduction

The word "dynamics" is one of the most used in mathematics. Here we use it in the sense of *local discrete holomorphic dynamics*, namely, the study of iterates of a germ of a holomorphic map in $\mathbb{C}^n$, $n \geq 1$ near a fixed point. Aside from its own interest, the study of such dynamics is useful to understand global dynamics of foliations or vector fields (considering the germ as the holonomy on a compact leaf). Although the germ might be non-invertible, here we will concentrate only on holomorphic diffeomorphisms.

Let F denote such a germ of holomorphic diffeomorphism in a neighborhood of the origin 0 in $\mathbb{C}^n$. As expected, the dynamical behavior of the sequence of iterates $\{F^{\circ q}\}_{q \in \mathbb{N}}$ of F in a neighborhood of 0 is described at the first order by the dynamics of its differential dF_0. In fact, depending on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of dF_0, in some cases both dynamics are the same.

The so-called "hyperbolic case" is the generic case, that is, when none of the eigenvalues is of modulus 1. In this case the map is topologically conjugate to its differential (by the Hartman-Grobman theorem [21,22,27]) and the dynamics is then completely clear. In case the eigenvalues have either all modulus strictly smaller than one or all strictly greater than one, then the origin is an attracting or respectively repelling fixed point for an open neighborhood of 0. Also, by the stable/unstable manifold theorem, there exists a holomorphic (germ of) manifold invariant under F and tangent to the sum of the eigenspaces of those λ_j's such that $|\lambda_j| < 1$ (*respectively* $|\lambda_j| > 1$) which is attracted to (*respectively* repelled from) 0. However, already in case when all eigenvalues have modulus different from 1, holomorphic linearization is not always possible due to the presence of resonances among the eigenvalues (see, for instance, [7, Chapter IV]).

The "non-hyperbolic case" is the most interesting from a dynamical point of view. In dimension one, $F(z) = \lambda z + \dots$ and $|\lambda| = 1$, the dynamics depends on the arithmetic properties of λ. Namely, if λ is a root of 1 (the so called "parabolic case") then either F is linearizable (which is the case if and only if $F^{\circ m} = \mathsf{id}$ for some $m \in \mathbb{N}$) or there exist certain F-invariant sets, called "petals", which form a pointed neighborhood of 0 and which are alternatively attracting and repelling (and permuted each other by the multiplicity of λ as a root of unity). On such petals the map F (or F^{-1}) is conjugate to an Abel translation of the type $z \mapsto z+1$ via a change of coordinates which is nowadays known as "Fatou coordinates". This is the content of the famous Leau-Fatou flower theorem, which we will recall in detail in Section 2. In such a parabolic (non linearizable) case, the topological classification is rather simple (see [14] and [28]), in fact, the map is topologically equivalent to $z \mapsto \lambda z(1 + z^{mk})$, where $\lambda^m = 1$. While, from the formal point of view, the map F is conjugate to $z \mapsto \lambda z + z^{mk+1} + az^{2mk+1}$ for some "index" $a \in \mathbb{C}$ which can be computed as a residue around the origin. The holomorphic classification is however much more complicated and it is due to Voronin [41] and Écalle [17, 18]. The very rough idea for germs tangent to identity is to consider the changes of Fatou coordinates on the intersection between an attracting petal and the subsequent repelling petal. This provides twice the multiplicity of F of certain holomorphic functions which are known as "sectorial invariants". These invariants, together with the multiplicity and the index, are the sought complete system of holomorphic invariants.

In case λ has modulus one but it is not a root of unity, the map is called "elliptic". In such a case the germ is always formally linearizable, but, as strange as it might be, it is holomorphically linearizable if and only if topologically linearizable (and this last condition is related to boundedness of the orbits in a neighborhood of 0). Writing $\lambda = e^{2\pi i\theta}$, first Siegel and later Bruno and Yoccoz [42] showed that holomorphic linearization depends on the arithmetic properties of $\theta \in \mathbb{R}$. In particular they showed that for almost every $\theta \in \mathbb{R}$ the germ is holomorphically linearizable. Later Yoccoz proved that the arithmetic condition for which every map starting with $e^{2\pi i\theta}z + \dots$ is linearizable can be characterized exactly (in the sequel we will refer to such a condition as the "Bruno condition", but we are not going to write it here explicitly). In particular, the quadratic polynomial $e^{2\pi i\theta}z+z^2$ is holomorphically linearizable if and only if θ satisfies the Bruno condition. Non-linearizable elliptic germs present very interesting dynamics that we are not going to describe in details here, leaving it to the reader to check the survey papers [2, 8, 9].

In higher dimension, the situation is much more complicated. To be precise, only the definition of "hyperbolic germs" makes really sense. There is not such a clear distinction between parabolic or elliptic germs. And, as we will try to make clear in these notes, this is not just a matter of definition, but it is really a matter of dynamics that, in higher dimension, can mix different types of behaviors without privileging none. We will concentrate on the parabolic behavior. And we will see how, even germs which one would call "elliptic" can have a parabolic attitude.

The first instance of parabolic behavior in higher dimension is clearly a map tangent to the identity. This is the prototype of parabolic dynamics. It has been proved by Écalle [18] and Hakim [26] that generically there exist "petals", also called "parabolic curves", namely, one-dimensional F-invariant analytic discs having the origin in their boundary and on which the dynamics is of "parabolic type", namely, the restriction of the map is a Abel type translation. Later, Abate [1] (see also [3]) proved that such petals always exist in dimension two. Hakim also gave conditions for which the petals are "fat" in the sense that there exist basins of attraction modeled on such parabolic curves. We will describe such results in details in Section 3.

Other examples of parabolic behaviors are when one eigenvalue is 1. However, in such a case it is not always clear that some "parabolic attitude" exists, depending on the other eigenvalues and some invariants. Hakim [25] (based on the previous work by Fatou [19] and Ueda [39, 40] in $\mathbb{C}^2$) studied the *semi-attractive* case, with one eigenvalue equal to 1 and the rest of eigenvalues having modulus less than 1. She proved that either there exists a curve of fixed points or there exist attracting open petals, modeled on parabolic curves. Such a result has been later generalized by Rivi [33] and Rong [37].

The case when one eigenvalue is 1 and the other has modulus equal to one but is not a root of unity has been studied in [10] – the so-called "quasi-parabolic" case – and it has been proved that, under a certain generic hypothesis called "dynamical separation", there exist petals tangent to the eigenspace of 1, so that, in such a case, there is a parabolic attitude. Such a result has been generalized to higher dimension by Rong [34–36]. We will describe more in details such results in Section 4.

However, as recently proved in [12], parabolic behavior can appear, maybe unexpected, also in those situations when no eigenvalue is a root of unity. Indeed, the new phenomenon, which generates "parabolic attitude" discovered in [12] can be roughly summarized as follows. Assume for simplicity that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the differential dF_0 have a unique one-dimensional resonance of type $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} = 1$. Let $\hat{F}$ be the formal Poincaré-Dulac normal form of F. For the moment, assume that $\hat{F}$ and the formal conjugation are converging. Then $\hat{F}$ has an invariant one-codimensional foliation given by $\mathcal{F} := \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \text{const}\}$ (and so does F). Considering the map $\varphi : \mathbb{C}^n \to \mathbb{C}$ given by $\varphi(z_1, \ldots, z_n) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, it follows that $\varphi \circ \hat{F}(z) = f(\varphi(z))$, where f is a germ in $\mathbb{C}$ tangent to the identity. In other words, $\hat{F}$ acts as a germ tangent to the identity on the space of leaves of the foliation $\mathcal{F}$. The idea is then that the parabolic dynamics (petals) on such a space can be pulled back to $\mathbb{C}^n$ and creates invariant sets that, under some suitable conditions, are basins of attraction for $\hat{F}$ (and thus for F). In fact, the similar argument is correct also when the conjugation of F to its formal Poincaré-Dulac normal form is not converging, although much more complicated. We will describe more in details that in Section 5, where we also provide explicit examples.

A final warning about the paper. These notes are by no means intended to be a survey paper on parabolic dynamics in higher dimensions, so that we are not going to cite all the results proved in this direction so far: for this, we refer the readers to the papers [8] and, mainly, [2]. These notes are intended to be a hint of what the word "parabolic" should mean in higher dimensional local holomorphic dynamics.

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2 Parabolic dynamics in one variable

Definition 2.1. Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. Let $\mu = a_{k+1}/|a_{k+1}|$ and $v \in \partial\mathbb{D}$; we say that v is an *attracting direction* if $\mu v^k = -1$, we say that v is a *repelling direction* if $\mu v^k = 1$.

Clearly there exist exactly k attracting and k repelling directions.

Remark 2.2. The attracting directions of f are the repelling directions of f^{-1} and conversely the repelling directions of f are the attracting directions of f^{-1}.

Definition 2.3. Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. An *attracting petal* centered at an attracting direction v is a simply connected open set P_v such that

(1) $O \in \partial P_v$;
(2) $f(P_v) \subseteq P_v$;
(3) $\lim_{n\to\infty} f^{\circ n}(z) = O$ and $\lim_{n\to\infty} \frac{f^{\circ n}(z)}{|f^{\circ n}(z)|} = v$ for all $z \in P_v$.

A *repelling petal* centered at a repelling direction v is an attracting petal for f^{-1} centered at the attracting direction v (for f^{-1}).

As a matter of notation, let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. We write $v_1^+, \ldots, v_k^+$ for the attracting directions of f and $v_1^-, \ldots, v_k^-$ for the repelling directions of f, ordered so that starting from 1 and moving counterclockwise on $\partial\mathbb{D}$ the first point we meet is v_1^+, then v_1^-, then v_2^+ and so on.

Here we state the following version of the Leau-Fatou flower theorem:

Theorem 2.4 (Leau-Fatou). *Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. Let $\{v_1^+, \ldots, v_k^+, v_1^-, \ldots, v_k^-\}$ be the ordered attracting and repelling directions of f. Then:*

(1) *for any $v_j^\pm$ there exists an attracting/repelling petal $P_{v_j^\pm}$ centered at $v_j^\pm$;*
(2) *$P_{v_j^+} \cap P_{v_l^+} = \emptyset$ and $P_{v_j^-} \cap P_{v_l^-} = \emptyset$ for $j \neq l$;*
(3) *for any attracting petal $P_{v_j^+}$ the function $f|_{P_{v_j^+}}$ is holomorphically conjugate to $\zeta \mapsto \zeta + 1$ defined on $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > C\}$ for some $C > 0$;*
(4) *$|f^{\circ m}(z)|^k \sim \frac{1}{m}$ for all $z \in P_{v_j^+}$, $j = 1, \ldots, k$.*

Proof. The proof can be found, *e.g.* in [16] or [29]. We only mention here how (4) is obtained. Up to a dilation one can assume $a_{k+1} = -1/k$ and $v_1^+ = 1$. By the Leau-Fatou construction, if $C > 0$ is sufficiently large, then setting $H := \{w \in \mathbb{C} : \operatorname{Re} w > C\}$, $\Psi(w) := w^{-1/k}$ for $w \in H$ with the k-th root chosen such that $1^{1/k} = 1$ and $P := \Psi(H)$, the conjugate map $\varphi := \Psi^{-1} \circ f \circ \Psi : H \to H$ satisfies

$$\varphi(w) = w + 1 + O(|w|^{-1}), \quad w \in H.$$

From this, (4) follows at once. $\square$

3 Germs tangent to the identity

Definition 3.1. Let F be a germ of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let $F(X) = X + P_h(X) + \ldots$, $h \geq 2$ be the expansion of F in homogeneous polynomials, $P_h(X) \neq 0$. The polynomial $P_h(X)$ is called the *Hakim polynomial* and the integer h the *order* of F at O.

Let $v \in \mathbb{C}^n$ be a nonzero vector such that $P_h(v) = \alpha v$ for some $\alpha \in \mathbb{C}$. Then v is called a *characteristic direction* for F. If moreover $\alpha \neq 0$ then v is said a *nondegenerate characteristic direction*.

A *parabolic curve* for a map F tangent to the identity is a holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^n$ from the unit disc to $\mathbb{C}^n$, continuous up to the boundary and such that $F(\varphi(\mathbb{D})) \subset \varphi(\mathbb{D})$ and $F^{\circ m}(\varphi(\zeta)) \to 0$ for all $\zeta \in \mathbb{D}$. Moreover, the parabolic curve φ is tangent to a direction $v \in \mathbb{C}^n \setminus \{0\}$ if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}$ as $\mathbb{D} \ni \zeta \to 0$ (here $[v]$ denotes the class of v in $\mathbb{P}^{n-1}$).

It can be proved that if P is a parabolic curve for F at O tangent to v then v is a characteristic direction. However there exist examples of germs tangent to the identity with a parabolic curve not tangent to a single direction (that is with tangent cone spanning a vector space of dimension greater than one).

Theorem 3.2 (Écalle, Hakim). *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O with order h. If v is a nondegenerate characteristic direction for F then there exist* (*at least*) $h-1$ *parabolic curves tangent to v.*

Hakim's proof relies essentially on a finite number of blow-ups and changes of coordinates in such a way that the map assumes a good form and one can define an operator (which is a contraction) on a suitable space of curves. The fixed point of such an operator is the wanted curve.

Actually Hakim's work provides the existence of basins of attraction or lower dimensional invariant manifolds which are attracted to the origin, called *parabolic manifolds*, according to other invariants related to any nondegenerate characteristic direction. Let v be a nondegenerate characteristic direction for F and let P_h be the Hakim polynomial. We denote by $A(v) := d(P_h)_{[v]} - \mathrm{id} : T_{[v]}\mathbb{CP}^{n-1} \to T_{[v]}\mathbb{CP}^{n-1}$. Then we have

Theorem 3.3 (Hakim). *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let v be a nondegenerate characteristic direction. Let $\beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$ be the eigenvalues of $A(v)$. Moreover assume* $\mathsf{Re}\,\beta_1, \ldots, \mathsf{Re}\,\beta_m > 0$ *and* $\mathsf{Re}\,\beta_{m+1}, \ldots,$ $\mathsf{Re}\,\beta_{n-1} \leq 0$ *for some $m \leq n-1$ and let E be the sum of the eigenspaces associated to $\beta_1, \ldots, \beta_m$. Then there exists a parabolic manifold M of dimension $m+1$ tangent to $\mathbb{C}v \oplus E$ at O such that for all $p \in M$ the sequence $\{F^{\circ k}(p)\}$ tends to O along a trajectory tangent to v.*

In particular if all the eigenvalues of $A(v)$ have positive real part then there exists a basin of attraction for F at O.

In [1] Abate proved the following:

Theorem 3.4 (Abate). *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ having the origin as an isolated fixed point and tangent to the identity at O with order h. Then there exist (at least) $h-1$ parabolic curves.*

The original proof of Abate is rather involved. Other simpler or clearer proofs have been discovered later. In [3], a very general construction explaining the essence of the argument in Abate's proof has been developed. In [13] Brochero, Cano and Hernanz gave a proof of such a theorem which relies on foliations. Namely, they associate to the germ F a formal vector field X such that $\exp(X) = F$. Then they perform a number of blow-ups to "solve" the singularities of X using the Seidenberg theorem (which holds in the formal category) and then use the Camacho-Sad construction [15] proving that, on a smooth component of the exceptional divisor, there exists a suitable "good" singularity for X. Such a good singularity corresponds to the existence of a nondegenerate characteristic direction for the blow-up of F, hence applying Theorem 3.2 one gets parabolic curves that can be projected downstairs.

This method of taking a germ tangent to the identity and associating a formal vector field whose time one flow is the germ itself is a good way to transfer the result from the better known theory of vector fields to the study of germs tangent to the identity. However, there is a disadvantage with respect to more direct methods: since the vector field is in general only formal, deep problems of resurgence can occur. It is likely that one could prove Abate's theorem directly in the category of formal vector fields (namely, without using Theorem 3.2) by using resurgence to study a formal separatrix of the formal vector field associated with the map and obtain the parabolic curves by summation.

In the same direction, in the recent paper [6], Abate and Tovena studied *real* dynamics of *complex* homogeneous vector fields. Besides its intrinsic interest, this is an useful problem to study because the discrete dynamics of the time 1-map is encoded in the real integral curves of the vector field, and time 1-maps of homogeneous vector fields are prototypical examples of holomorphic maps tangent to the identity at the origin. The main idea here is that, roughly speaking, integral curves for homogeneous vector fields are geodesics for a meromorphic connection on a projective space.

More generally, thanks to a result by Takens [38] (see also [24, Chapter 1]), in case of diffeomorphisms with *unipotent* linear part, one can embed such germs in the flow of formal vector fields, so that this type of argument might be used also in such cases. On the other hand, when the linear part of the diffeomorphism is not unipotent, the author is not aware of any general result about embedding such a diffeomorphism into the flow of a formal vector field. In fact, one encounters somewhat unexpected differences between the dynamics of diffeomorphisms and that of vector fields, see Raissy [32].

In [3] (see also [4, 5, 11]) a different point of view, more abstract but more intrinsic, has been adopted. Blowing up the origin one obtains a new germs of diffeomorphism pointwise fixing the exceptional divisor. Thus the situation is that of considering a holomorphic self-map of a complex manifold having a (hyper)surface of fixed points. Roughly speaking, the differential of the map acts on the normal bundle of such a hypersurface in a natural way and thus creates a meromorphic connection, whose singularities essentially rule the dynamics of the map.

4 Semiattractive and quasi-parabolic germs

4.1 Semiattractive germs

We say that a parabolic germ F is *semi-attractive* if 1 is an eigenvalue of dF_O and all the other eigenvalues have modulus strictly less than 1 (if all the other eigenvalues have modulus strictly greater than 1 we argue on F^{-1}). There are essentially two cases to be distinguished here: either F has or has not a submanifold of fixed points.

In case F has a submanifold of fixed points (of the right dimension) there is a result due to Nishimura [30] which roughly speaking says that, in absence of resonances, F is conjugate along S to its action on the normal bundle N_S to S in $\mathbb{C}^n$.

In case F has no curves of fixed points, Hakim [25] (based on the previous work by Fatou [19] and Ueda [39, 40] in $\mathbb{C}^2$) proved that, under suitable generic hypotheses, there exist "fat petals" (called *parabolic manifolds* or *basins of attraction* when they have dimension n) for F at O. That is

Theorem 4.1 (Hakim). *Let F be a semi-attractive parabolic germ at O, with 1 as eigenvalue of dF_O of (algebraic) multiplicity 1. If O is an isolated fixed point of F then there exist k disjoint basins of attraction for F at O, where $k+1 \geq 2$ is the "order" of $F - \mathrm{id}$ at O.*

It is worth noticing that if F is an automorphism of $\mathbb{C}^2$ then each basin of attraction provided by Theorem 4.1 is biholomorphic to $\mathbb{C}^2$ (the existence of proper subsets of $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$ for $n > 1$ is known as the *Fatou-Bierbach phenomenon*).

Theorem 4.1 is a special case of the procedure described in the Introduction and in Section 5. Indeed, write the map F as

$$F(z,w)=(z+az^{k+1}+O(\|zw\|,\|w\|^2,|z|^{k+2}),\lambda w+O(\|z\|^2,\|zw\|,\|w\|^2),$$

where $w\in\mathbb{C}^{n-1}$, λ is a $(n-1)\times(n-1)$ matrix with eigenvalues of modulus strictly less than one, $k\in\mathbb{N}\cup\{\infty\}$ and $a\in\mathbb{C}\setminus\{0\}$. The case $k=\infty$ (namely, no pure terms in z are present in the first component) corresponds to the existence of a curve of fixed points. So we assume $k\in\mathbb{N}$. The monomials in w are not resonant in the first component, thus they can be killed by means of a Poincaré-Dulac formal change of coordinates. Hence, the foliation $\{z=\text{const}\}$ is invariant by the formal normal form of F, and the action on the "space of leaves" (which is nothing but $\mathbb{C}$) is given exactly by $z\mapsto z+az^{k+1}+\ldots$ If we perform the Poincaré-Dulac procedure solving only finitely many homological equations to kill terms in w in the first component, the map F has the form:

$$F(z,w)=(z+az^{k+1}+O(z^{k+2})+O(\|w\|^l),\lambda w+\ldots)$$

with $l\gg 1$ as big as we want. Hence, the map F does not preserve the foliation $\{z=\text{const}\}$, but it moves it slowly at order l in w. This implies that the preimage under the map $(z,w)\mapsto z$ of a (suitable) sector S contained in some petals of $z\mapsto z+az^{k+1}$ is in fact invariant for F. Taking open sets of the form $\{(z,w):\|w\|<|z|^\beta, z\in S\}$ for a suitable $\beta>0$, it can be then proved that such open sets are invariant and, via the dynamics "downstair", they are actually basins of attraction for F.

4.2 Quasi-parabolic germs

We call *quasi-parabolic* a germ if all eigenvalues of dF_O have modulus 1 and at least one, but not all of them, is a root of unity. Replacing the germ with one of its iterates, we can assume that all the eigenvalues which are roots of unity are 1.

Let us then write the spectrum of dF_O as the disjoint union $\{1\}\cup E$. In case the eigenvalues in E have no resonances and satisfy a Bruno-type condition, a result of Pöschel [31] assures the existence of a complex manifold M tangent to the eigenspace associated to E at O which is F-invariant and such that the restriction of F to M is holomorphically conjugate to the restriction of dF_O to this eigenspace.

We describe here the "parabolic attitude" of quasi-parabolic germs in $\mathbb{C}^2$. For results in $\mathbb{C}^n$ we refer to [34,35]. As strange as it may seem, it is not known whether all quasi-parabolic germs have "parabolic attitude"!

Using Poincaré-Dulac theory, since all resonances are of the type $(1,(m,0))$, $(2,(m,1))$ (namely, in the first coordinate the resonant monomials are just z^m and in the second coordinate the resonant monomials are $z^m w$), the map F is formally conjugate to a map of the form

$$\hat{F}(z,w) = \left(z + \sum_{j=\nu}^{\infty} a_j z^j, e^{2\pi i\theta} w + \sum_{j=\mu}^{\infty} b_j z^j w\right), \tag{4.1}$$

where we assume that either $a_\nu \neq 0$ or $\nu = \infty$ if $a_j = 0$ for all j. Similarly for b_μ.

As it is proved in [10], the number $\nu(F) := \nu$ is a formal invariant of F. Moreover, it is proved that, in case $\nu < +\infty$, the sign of $\Theta(F) := \nu - \mu - 1$ is a formal invariant. The map F is said *dynamically separating* if $\nu < +\infty$ and $\Theta(F) \leq 0$.

The next proposition is proved in [12] and its proof is a simple argument based on the implicit function theorem:

Proposition 4.2. *Let F be a quasi-parabolic germ of diffeomorphism of $\mathbb{C}^2$ at 0. Then $\nu(F) = +\infty$ if and only if there exists a germ of (holomorphic) curve through 0 that consists of fixed points of F.*

In case $\nu(F) < +\infty$, the following result is proved in [10]:

Theorem 4.3. *Let F be a quasi-parabolic germ of diffeomorphism of $\mathbb{C}^2$ at 0. If F is dynamically separating then there exist $\nu(F) - 1$ parabolic curves for F at 0.*

The argument in [10] is based on a series of blow-ups and changes of coordinates which allow to write F into a suitable form so that one can write a similar operator to the one defined by Hakim and prove that its fixed points in a certain Banach space of curves are exactly the sought parabolic curves.

As in the semi-attractive case, one can argue using the invariant formal foliation $\{z = \text{const}\}$. Contractiveness in the w-variable is however not for free here. Indeed, in [12] (see also Section 5) it is proved

Proposition 4.4. *Let F be a dynamically separating quasi-parabolic germ, formally conjugate to* (4.1). *If*

$$\mathsf{Re}\left(\frac{b_{\nu-1}}{e^{2\pi i\theta} a_\nu}\right) > 0,$$

then there exist $\nu(F) - 1$ disjoint connected basins of attraction for F at 0.

It should be remarked that the condition in the previous proposition is destroyed under blow-ups.

The non-dynamically separating case is still open. In such a case there is still a formal foliation $\{z = \mathrm{const}\}$ which is invariant, but the w-variable cannot be controlled appropriately by the z variable.

F. Fauvet [20] told me that using Écalle's resurgence theory it is possible to prove that parabolic curves exist also in the non-dynamically separating case when the other eigenvalue satisfies a Bruno-type condition.

5 One-resonant germs

Let F be a germ of holomorphic diffeomorphism in $\mathbb{C}^n$ fixing 0. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the linear part. We say that F is *one-resonant with respect to the first m eigenvalues* $\{\lambda_1, \dots, \lambda_m\}$ $(1 \leq m \leq n)$ (or *partially one-resonant*) if there exists a fixed multi-index $\alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0) \neq 0 \in \mathbb{N}^n$ such that for $s \leq m$, the resonances $\lambda_s = \prod_{j=1}^n \lambda_j^{\beta_j}$ are precisely of the form $\lambda_s = \lambda_s \prod_{j=1}^m \lambda_j^{k\alpha_j}$, where $k \geq 1 \in \mathbb{N}$ is arbitrary.

This notion has been introduced in [12]. The main advantage of such a notion of partial one-resonance is that it can be applied to the subset of all eigenvalues of modulus equal to 1, regardless of the relations which might occur among the other eigenvalues.

In case of partial one-resonance, the classical Poincaré-Dulac theory implies that, whenever F is not formally linearizable in the first m components, F is formally conjugate to a map whose first m components are of the form $\lambda_j z_j + a_j z^{\alpha k} z_j + R_j(z)$, $j = 1, \dots, m$, where, the number $k \in \mathbb{N}$ is an invariant, called *the order of F with respect to* $\{\lambda_1, \dots, \lambda_m\}$, the vector $(a_1, \dots, a_m) \neq 0$ is invariant up to a scalar multiple and the R_j's contain only resonant terms. The fact that the number

$$\Lambda(F) := \sum_{j=1}^m \frac{a_j}{\lambda_j} \alpha_j$$

is equal or not to zero is an invariant, and the map F is said to be *non-degenerate* provided $\Lambda(F) \neq 0$. In fact, one can always rescale the map to make $\Lambda(F) = 1$ provided it is not zero.

In [12] it is proved that a partially one-resonant non-degenerate germ F has a simple formal normal form $\hat{F}$ such that

$$\hat{F}_j(z) = \lambda_j z_j + a_j z^{k\alpha} z_j + \mu \frac{\alpha_j}{\lambda_j} z^{2k\alpha} z_j, \quad j = 1, \dots, m.$$

Although none of the eigenvalues λ_j, $j = 1, \dots, m$, might be roots of unity, such a normal form is the exact analogue of the formal normal form for parabolic germs in $\mathbb{C}$. In fact, a one-resonant germ acts as a parabolic germ on the space of leaves of the formal invariant foliation $\{z^\alpha = \text{const}\}$ and that is the reason for this parabolic-like behavior.

Let F be a one-resonant non-degenerate diffeomorphism with respect to the eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. We say that F is *parabolically attracting* with respect to $\{\lambda_1, \ldots, \lambda_m\}$ if

$$|\lambda_j| = 1, \quad \mathsf{Re}\left(\frac{a_j}{\lambda_j}\frac{1}{\Lambda(F)}\right) > 0, \quad j = 1, \ldots, m.$$

Again, such a condition is invariant, indeed, conjugating the map, both $\Lambda(F)$ and the a_j's vary suitably and the sign of the previous expression is unchanged. Such a condition is vacuous in dimension 1 or whenever $m = 1$ since in that case $\alpha = (\alpha_1, 0, \ldots, 0)$ with $\alpha_1 > 0$. In [12] it is proved the following:

Theorem 5.1. *Let F be a holomorphic diffeomorphism germ at 0 that is one-resonant, non-degenerate and parabolically attracting with respect to $\{\lambda_1, \ldots, \lambda_m\}$. Suppose that $|\lambda_j| < 1$ for $j > m$. Let $k \in \mathbb{N}$ be the order of F with respect to $\{\lambda_1, \ldots, \lambda_m\}$. Then F has k disjoint basins of attraction having 0 on the boundary.*

The different basins of attraction for F (that may or may not be connected) project via the map $z \mapsto u = z^\alpha$ into different petals of the germ $u \mapsto u + \Lambda(F)u^{k+1}$.

A semi-attractive germ is always one-resonant, non-degenerate and parabolically attracting, thus the previous theorem is a generalization of Hakim's result.

It should be noticed that the conditions about non-degeneracy and parabolically attractiveness are sharp. In [12] examples are given of one-resonant germs for which such conditions are not satisfied and that have no basins of attractions.

More interesting for the purpose of these notes is the following example (still from [12]). Let $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$F(z, w) = (\lambda z + az^2 w + \ldots, \lambda^{-1} w + bzw^2 + \ldots),$$

with $|a| = |b| = 1$. Then F is one-resonant with index of resonance $(1, 1)$ and for each choice of (a, b) such that the germ is non-degenerate (*i.e.* $a\lambda^{-1} + b\lambda \neq 0$), there exists a basin of attraction for F at 0. Indeed, it can be checked that the non-degeneracy condition implies that F is parabolically attracting with respect to $\{\lambda, \lambda^{-1}\}$ and hence Theorem 5.1 applies.

A similar argument can be applied to F^{-1}, producing a basin of repulsion for F at 0. Hence we have a parabolic type dynamics for F.

On the other hand, suppose further that θ satisfies a Bruno condition. Since $\lambda^q \neq \lambda$ for all $q \in \mathbb{N}$, it follows from Pöschel's theorem [31, Theorem 1] that there exist two analytic discs through 0, tangent to the z-axis and to the w-axis respectively, which are F-invariant and such that the restriction of F on each such a disc is conjugate to $\zeta \mapsto \lambda\zeta$ or $\zeta \mapsto \lambda^{-1}\zeta$ respectively. Thus, in such a case, the elliptic and parabolic dynamics mix, although the spectrum of dF_0 is only of elliptic type.

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Power series with sum-product Taylor coefficients and their resurgence algebra

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Abstract. The present paper is devoted to power series of SP type, *i.e.* with coefficients that are syntactically sum-product combinations. Apart from their applications to analytic knot theory and the so-called "Volume Conjecture", SP-series are interesting in their own right, on at least four counts: (i) they generate quite distinctive resurgence algebras (ii) they are one of those relatively rare instances when the resurgence properties have to be derived directly from the Taylor coefficients (iii) some of them produce singularities that unexpectedly verify finite-order differential equations (iv) all of them are best handled with the help of two remarkable, infinite-order integral-differential transforms, mir and nir.

Contents

1 Introduction

1.1 Power series with coefficients of sum-product type

The notion of SP series

Sum-product series (or *SP*-series for short) are Taylor series:

$$j(\zeta) := \sum_{n\geq 0} J(n)\,\zeta^n \tag{1.1}$$

whose coefficients are syntactically of sum-product (SP) type:

$$\begin{aligned} J(n) &:= \sum_{\epsilon\leq m<n} \prod_{\epsilon\leq k\leq m} F\left(\frac{k}{n}\right) \\ &= \sum_{\epsilon\leq m<n} \exp\left(-\sum_{\epsilon\leq k\leq m} f\left(\frac{k}{n}\right)\right) \quad (\epsilon\in\{0,1\}). \end{aligned} \tag{1.2}$$

Summation starts at $\epsilon = 0$ unless $F(0) \in \{0, \infty\}$, in which case it starts at $\epsilon = 1$. It always ends at $n-1$, not n.[1] The two driving functions F

[1] This choice is to ensure near-invariance under the change $F(x) \mapsto 1/F(1-x)$. See Section 3.5.

and f are connected under $F \equiv \exp(-f)$. Unless stated otherwise, F will be assumed to be meromorphic, and special attention shall be paid to the case when F has neither zeros nor poles, *i.e.* when f is holomorphic.

The importance of SP-series comes from their analytic properties (isolated singularities of a quite distinctive type) and their frequent occurence in various fields of mathematics (ODEs, knot theory etc.).

As for the above definition, it is less arbitrary than may seem at first sight. Indeed, none of the following changes:

(i) changing the grid $\{k/n\}$ to $\{\mathrm{Const}\, k/n\}$
(ii) changing the lower summation bounds from 0 or 1 to 2,3 ...
(iii) changing the upper summation bound from $n-1$ to n or $n-2$, $n-3$ etc. or a multiple thereof
(iv) replacing the 0-accumulating products $\prod F(\frac{k}{n})$ by 1-accumulating products $\prod F(\frac{n-k}{n})$

– none of these changes, we claim, would make much difference or even (allowing for minor adjustments) take us beyond the class of SP-series.

Special cases of SP series

For F a polynomial or rational function (respectively a trigonometric polynomial) and for Taylor coefficients $J(n)$ defined by pure products $\prod$ (rather than sum-products $\sum\prod$) the series $j(\zeta)$ would be of hypergeometric (respectively q-hypergeometric) type. Thus the theory of SP-series extends – and bridges – two important fields. But it covers wider ground. In fact, the main impulse for developping it came from knot theory, and we didn't get involved in the subject until Stavros Garoufalidis[2] and Ovidiu Costin[3] drew our attention to its potential.

Overview

In this first paper, halfway between survey and full treatment[4], we shall attempt five things:

(i) bring out the *main analytic features* of SP-series, such as the dichotomy between their two types of singularities (*outer*/*inner*), and produce *complete systems of resurgence equations*, which encode in compact form the whole Riemann surface structure;

[2] An expert in knot theory who visited Orsay in the fall of 2006.

[3] An analyst who together with S. Garoufalidis has been pursuing an approach to the subject parallel to ours, but distinct: for a comparison, see Section 12.1.

[4] Two follow-up investigations [15, 16] are being planned.

(ii) *localise* and *formalise* the problem, *i.e.* break it down into the separate study of a number of *local* singularities, each of which is produced by a specific *non-linear functional transform* capable of a full analytical description, which reduces everything to *formal* manipulations on power series;
(iii) sketch the *general picture* for arbitrary driving functions F and f – pending a future, detailed investigation;
(iv) show that in many instances (f polynomial, F monomial or even just rational) our local singularities satisfy ordinary differential equations, but of a very distinctive type, which accounts for the "rigidity" of their resurgence equations, *i.e.* the occurrence in them of essentially *discrete* Stokes constants;[5]
(v) sketch numerous examples and then give a careful treatment, theoretical *and* numerical, of *one special case* chosen for its didactic value (it illustrates all the main SP-phenomena) and its practical relevance to knot theory (specifically, to the knot 4_1).

1.2 The outer/inner dichotomy and the ingress factor

The outer/inner dichotomy

Under analytic continuation, SP-series give rise to two distinct types of *singularities*, also referred to as *generators*, since under alien derivation they generate the resurgence algebra of our SP-series. On the one hand, we have the *outer generators*, so-called because they never recur under alien derivation (but produce inner generators), and on the other hand we have the *inner generators*, so-called because they recur indefinitely under alien derivation (but never re-produce the outer generators). These two are, by any account, the main types of generators, but for completeness we add two further classes: the *original generators* (*i.e.* the SP-series themselves) and the *exceptional generators*, which don't occur naturally, but can prove useful as auxiliary adjuncts.

A gratifying surprise: the *mir*-transform

We shall see that *outer* generators can be viewed as infinite sums of *inner generators*, and that the latter can be constructed quite explicitly by subjecting the driving function F to a chain of nine local transforms, all of which are elementary, save for one crucial step: the *mir*-transform. Furthermore, this *mir*-transform , though resulting from an unpromising mix

[5] Contrary to the usual situation, where these Stokes or resurgence constants are free to vary continuously.

of complex operations[6], will turn out to be an *integro-differential operator*, of infinite order but with a transparent expression that sheds much light on its analytic properties. We regard this fascinating *mir*-transform, popping out of nowhere yet highly helpful, as the centre-piece of this investigation.

The ingress factor and the cleansing of SP-series
Actually, rather than directly considering the SP-series $j(\zeta)$ with coefficients $J(n)$, it shall prove expedient to study the slightly modified series $j^{\#}(\zeta)$ with coefficients $J^{\#}(n)$ obtained after division by a suitably defined *ingress factor* $Ig_F(n)$ of strictly local character:

$$j^{\#}(\zeta) = \sum J^{\#}(n)\, \zeta^n \qquad \text{with} \qquad J^{\#}(n) := J(n)/Ig_F(n). \qquad (1.3)$$

This purely technical trick involves no loss of information[7] and achieves two things:

(i) the various *outer* and *inner* generators will now appear as purely *local* transforms of the driving function F viewed as an analytic germ at 0 or at some other suitable base point x_0 (in $[0, 1]$ or even outside);
(ii) distinct series $j_{F_i}(\zeta)$ relative to distinct base points x_i (or, put another way, to distinct translates $F_i(x) := F(x + x_i)$ of the same driving function) will lead to exactly the same *inner generators* and so to the same *inner algebra* – which wouldn't be the case but for the pre-emptive removal of Ig_F.

In any case, as we shall see, the ingress factor is a relatively innocuous function and (even when it is divergent-resurgent, as may happen) the effect not only of *removing it* but also, if we so wish, of *putting it back* can be completely mastered.

1.3 The four gates to the inner algebra

We have just described the various types of singularities or "generators" we are liable to encounter when analytically continuing a SP-series. Amongst these, as we saw, the *inner generators* stand out. They span the *inner algebra*, which is the problem's hard, invariant core. Let us now review the situation once again, but from another angle, by asking: how many *gates* are there for entering the unique *inner algebra*? There are, in effect, four types:

[6] Two Laplace transforms, direct and inverse, with a few violently non-linear operations thrown in.

[7] Since information about $j^{\#}$ immediately translates into information about j, and vice versa.

Gates of type 1: original generators. We may of course enter through an *original generator*, *i.e.* through a *SP series*, relative to any base point x_0 of our choosing. Provided we remove the corresponding ingress factor, we shall always arrive at the same inner algebra.

Gates of type 2: outer generators. We may enter through an *outer generator*, *i.e.* through the mechanism of the nine-link chain of Section 5, again relative to any base point. Still, when F does have zeros x_i, these qualify as privileged base points, since in that case we can make do with the simpler four-link chain of Section 5.

Gates of type 3: inner generators. We may enter through an *inner generator*, *i.e.* via the mechanism of the nine-link chain of Section 4, but only from a base point x_i where f (not F !) vanishes. By so doing, we do not properly speaking *enter* the inner algebra, but rather start *right there*. Due to the ping-pong phenomenon, this inner generator then generates all the other ones. The method, though, has the drawback of introducing a jarring dissymmetry, by giving precedence to *one* inner generator over all others.

Gates of type 4: exceptional generators. We may enter through a *exceptional* or *"mobile" generator*, *i.e.* once again via the mechanism of the nine-link chain of Section 4, but relative to any base point x_0 where f doesn't vanish.[8] It turns out that any such "exceptional generator" generates all the inner generators (– and what's more, *symmetrically* so –), but isn't generated by them. In other words, it gracefully *self-eliminates*, thereby atoning for its parasitical character. Exceptional generators, being "mobile", have the added advantage that their base point x_0 can be taken arbitrarily close to the base point x_i of any given inner generator, which fact proves quite helpful, computationally and also theoretically.

ACKNOWLEDGEMENTS. Since our interest in knot-connected power series (*i.e.* the series $G^{NP}_{\mathcal{K}}$ and $G^{P}_{\mathcal{K}}$ associated with a knot $\mathcal{K}$: *cf.* Section 9.1) and the closely related notion of SP-series (a natural and conceptually more appealing generalisation, in terms of which we chose to reframe the problem) was first awakened after the 2006 visit to Orsay of Stavros Garoufalidis and his pioneering joint work with Ovidiu Costin

[8] So that the so-called *tangency order* is $\kappa = 0$, whereas for the inner generators it is ≥ 1 and generically $= 1$. See Section 4.

and since, despite tackling the problem from very different angles, we have been keeping in touch for about one year, comparing methods and results, we feel we owe it to the reader to outline the main differences between our two approaches – to justify, as it were, their parallel existence.

The very first step is the same in both cases: we all rely on a quite natural method[9] for deducing the shape of a function's closest singularity, or singularities, from the exact asymptotics of its Taylor coefficients at 0.[10]

But then comes the question of handling the other singularities – those farther afield – and this is where our approaches start diverging. In [3–6], the idea is to re-write the functions under investigation in the form of multiple integrals amenable to the Riemann-Hilbert theory and then use the well-oiled machinery that goes with that theory. In this approach, the global picture (exact location of the singularities on the various Riemann leaves, rough nature of these singularities etc.) emerges first, and the exact description of each singularity, while also achievable at the cost of some extra work, comes second.

Our own approach reverses this sequence: the local aspect takes precedence, and we then piece the global picture together from the local data. To that end, we distinguish three types of "resurgence generators" (*i.e.* local singularities that generate the resurgence algebra under alien derivation): the actually occuring *inner* and *outer* generators[11], and the auxiliary *exceptional* or *movable* generators. The basic object here is the *inner resurgence algebra*, spanned by the *inner* generators, which recur indefinitely under alien derivation. The *outer* generators, on the other hand, produce only *inner* ones under alien derivation.[12] We give exact descriptions of both the *inner* and *outer* generators by means of special integro-differential functionals of infinite order: *nir*, *mir* and *nur*, *mur*.

[9] See O. C. and Section 2.3 of the present paper.

[10] This convergence is hardly surprising: the functions on hand (knot-related or SP) tend to verify no useable equations, whether differential or functional, that might give us a handle on their analytic properties, and so the Taylor coefficients are all we have to go by. Two of us (O. C. in [2] and J.E. in a 1993 letter to prof. G. K. Immink) hit independently on the same method – which must also have occurred, time and again, more or less explicitly, to many an analyst grappling with singularities.

[11] While there are only two *outer* generators (which may coalesce into one), there can be any number of *inner* generators.

[12] Which is only natural, since the *outer* generators can be interpreted as infinite sums of (self-reproducing) *inner* generators.

So much for the local aspect. To arrive at the global picture, we resort to an auxiliary construct, the so-called *exceptional* or *movable* generators, which are very useful on account of three features:

(i) they depend on a arbitrary *base point*, which can be taken as close as we wish to any particular singularity we want to zoom in onto;[13]
(ii) their own set of singularities include all the *inner* generators of the SP function;
(iii) they may also possess *parasitical* singularities[14] (*i.e.* singularities other than the above), but these always lie farther away from the *base point* than the closest *inner* generators.
Thus, by moving the base point around, we can reduce the global investigation to a local, or should we say, semi-local one, and derive the full picture, beginning with the crucial *inner algebra*.

A further difference between our approaches is this: while O. Costin and S. Garoufalidis are more directly concerned with the knot-related series $G_{\mathcal{K}}^{NP}$ and $G_{\mathcal{K}}^{P}$ and the so-called *volume conjecture* which looms ominously over the whole field, the framework we have chosen for our investigation is that of SP-series, *i.e.* general Taylor series with coefficients that are syntactically of sum-product type. But this latter difference might well be less than appears, since each of the two methods would seem, in principle, to be capable of extension in both directions.

2 Some resurgence background

2.1 Resurgent functions and their three models

The four models: formal, geometric, upper/lower convolutive
Resurgent "functions" exist in three/four types of models:

(i) The *formal model*, consisting of formal power series $\tilde{\varphi}(z)$ of a variable $z \sim \infty$. The tilda points to the quality of being "formal", *i.e.* possibly divergent;
(ii) the *geometric models* of direction θ. They consist of sectorial analytic germs $\varphi_\theta(z)$ of the same variable $z \sim \infty$, defined on sectors of aperture $> \pi$ and bisected by the axis $\arg(z^{-1}) = \theta$;

[13] Thus bringing it within the purview of the method of Taylor coefficient asymptotics (see above).

[14] This is the case *iff* the driving function F has at least one zero or one pole.

(iii) the *convolutive model*, consisting of *global microfunctions* of a variable $\zeta \sim 0$. Each microfunction possesses one *minor* (exactly defined, but with some *information missing*) and many *majors* (defined up to regular germs at the origin, *i.e.* with some *redundant information*). However, under a frequently fulfilled integrability condition at $\zeta \sim 0$, the *minor* contains all the information, *i.e.* fully determines the microfunction, in which case all calculations reduce to manipulations on the sole minors. As for the *globalness* of our microfunctions, it means that their minors possess the property of "endless analytic continuation": they can be continued analytically in the ζ-plane along any given (self-avoiding or self-intersecting, whole or punctured[15]) broken line starting from 0 and ending anywhere we like.

Usually, one makes do with a single convolutive model, but here it will be convenient to adduce two of them: the *upper* and *lower* models. In both, the minor-major relation has the same form ;

$$\begin{array}{llcl} & \text{minor} & & \text{major} \\ \text{upper} & \overset{\frown}{\varphi}(\zeta) & \equiv & \dfrac{1}{2\pi i}\Big(\overset{\smile}{\varphi}(\zeta\, e^{-\pi i}) - \overset{\smile}{\varphi}(\zeta\, e^{\pi i})\Big) \\ \text{lower} & \overset{\wedge}{\varphi}(\zeta) & \equiv & \dfrac{1}{2\pi i}\Big(\overset{\vee}{\varphi}(\zeta\, e^{-\pi i}) - \overset{\vee}{\varphi}(\zeta\, e^{\pi i})\Big) \end{array}$$

while the upper-lower correspondence goes like this:

$$\overset{\wedge}{\varphi}(\zeta) \equiv \partial_\zeta\, \overset{\frown}{\varphi}(\zeta) \quad ; \quad \overset{\vee}{\varphi}(\zeta) \equiv -\partial_\zeta\, \overset{\smile}{\varphi}(\zeta). \tag{2.1}$$

One of the points of resurgent analysis is to resum divergent series of "natural origin", *i.e.* to go from the formal model to the geometric one via one of the two convolutive models. Concretely, we go from *formal* to *convolutive* by means of a formal or term-wise Borel transform[16] and from *convolutive* to *geometric* by means of a θ-polarised Laplace trans-

[15] The broken line may be punctured at a finite number of singularity-carrying points $\zeta_1, \ldots, \zeta_n$, in which case we demand analytic continuability along *all* the 2^n paths that follow the broken line but circumvent each ζ_i to the right or to the left.

[16] Thus *upper* (respectively *lower*) Borel takes $\sum a_n\, z^{-n}$ to $\sum a_n\, \frac{\zeta^n}{n!}$ $\left(\text{respectively } \sum a_n\, \frac{\zeta^{n-1}}{(n-1)!}\right)$.

form, *i.e.* with integration along the half-axis $\arg(\zeta) = \theta$.

	upper	lower
geometric	$\varphi_\theta \xleftarrow{\overline{\mathcal{L}}_\theta} \overset{0}{\varphi}$	$\varphi_\theta \xleftarrow{\underline{\mathcal{L}}_\theta} \overset{\diamond}{\varphi}$
convolutive	$\overset{0}{\varphi}$	$\overset{\diamond}{\varphi}$
formal	$\tilde{\varphi} \xrightarrow{\overline{\mathcal{B}}} \overset{0}{\varphi}$	$\tilde{\varphi} \xrightarrow{\underline{\mathcal{B}}} \overset{\diamond}{\varphi}$

Resurgent algebras: the multiplicative structure

Resurgent functions are stable not just under *addition* (which has the same form in all models) but also under a *product* whose shape varies from model to model:

(i) in the *formal* model, it is the ordinary multiplication of power series;
(ii) in the *geometric* model, it is the pointwise multiplication of analytic germs;
(iii) in the *convolutive* models, it is the *upper/lower* convolution, with distinct expressions for minors[17] and majors:

minor convolution

upper $\overline{*}$ $$(\overset{\frown}{\varphi}_1 \,\overline{*}\, \overset{\frown}{\varphi}_2)(\zeta) := \int_0^\zeta \overset{\frown}{\varphi}_1(\zeta-\zeta_2)\, d\overset{\frown}{\varphi}_2(\zeta_2)$$

lower $\underline{*}$ $$(\overset{\wedge}{\varphi}_1 \,\underline{*}\, \overset{\wedge}{\varphi}_2)(\zeta) := \int_0^\zeta \overset{\wedge}{\varphi}_1(\zeta-\zeta_2)\, \overset{\wedge}{\varphi}_2(\zeta_2)\, d\zeta_2$$

major convolution

upper $\overline{*}$ $$(\overset{\smile}{\varphi}_1 \,\overline{*}\, \overset{\smile}{\varphi}_2)(\zeta) := \frac{1}{2\pi i}\int_{I(\zeta,u)} \overset{\smile}{\varphi}_1(\zeta-\zeta_2)\, d\overset{\smile}{\varphi}_2(\zeta_2)$$

lower $\underline{*}$ $$(\overset{\vee}{\varphi}_1 \,\underline{*}\, \overset{\vee}{\varphi}_2)(\zeta) := \frac{1}{2\pi i}\int_{I(\zeta,u)} \overset{\vee}{\varphi}_1(\zeta-\zeta_2)\, \overset{\vee}{\varphi}_2(\zeta_2)\, d\zeta_2$$

with $I(\zeta,u) = \left[\frac{1}{2}\zeta + e^{-\frac{\pi i}{2}}u, \frac{1}{2}\zeta + e^{+\frac{\pi i}{2}}u\right]$ and $0 < \frac{\zeta}{u} \ll 1$. For *major convolution* we *first* fix an auxiliary point u close enough to 0 (so as to

[17] Minor convolution is possible only under a suitable integrability condition at the origin. That condition is automatically met when *one* (hence *all*) majors verify $\overset{\smile}{\varphi}(\zeta) \to 0$ or $\zeta\, \overset{\vee}{\varphi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly on any sector of finite aperture.

steer clear of possible singularities in the convolution factors $\overset{\vee}{\varphi}_i$ or $\overset{\smile}{\varphi}_i$) and *then* calculate the convolution integral for ζ closer still to 0. The resulting integral does depend on u, but only up to a regular germ at 0, which doesn't affect the *class* of the convolution-major.

The upper/lower Borel-Laplace transforms

For simplicity, let us fix the polarisation $\theta = 0$ and drop the index θ in the geometric model $\varphi_\theta(z)$. We get the familiar formulas, reproduced here just for definiteness:

$$\begin{array}{cccc}
\text{multiplicative} & & & \text{convolutive} \\
 & \text{upper} & & \overset{0}{\varphi}(\zeta) = \{\overset{\frown}{\varphi}(\zeta), \overset{\smile}{\varphi}(\zeta)\} \\
 & \overline{\mathcal{B}} \nearrow \swarrow \overline{\mathcal{L}} & & \\
\varphi(z) & & & \downarrow \partial_\zeta \;\;,\;\; \downarrow -\partial_\zeta \\
 & \underline{\mathcal{B}} \searrow \nwarrow \underline{\mathcal{L}} & & \\
 & \text{lower} & & \overset{\Diamond}{\varphi}(\zeta) = \{\overset{\wedge}{\varphi}(\zeta), \overset{\vee}{\varphi}(\zeta)\}
\end{array}$$

$$\begin{array}{lll}
\text{minors} & \text{Borel} & \text{Laplace} \\
\text{upper} & \overset{\frown}{\varphi}(\zeta) = \frac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{\zeta z}\varphi(z)\frac{dz}{z} & \varphi(z) = \displaystyle\int_0^{+\infty} e^{-z\zeta}\, d\overset{\frown}{\varphi}(\zeta) \\
\text{lower} & \overset{\wedge}{\varphi}(\zeta) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{\zeta z}\varphi(z)dz & \varphi(z) = \displaystyle\int_0^{+\infty} e^{-z\zeta}\, \overset{\wedge}{\varphi}(\zeta)\, d\zeta \\
\text{majors} & \text{Laplace} & \text{Borel} \\
\text{upper} & \overset{\smile}{\varphi}(\zeta) = \displaystyle\int_c^{+\infty} e^{-\zeta z}\varphi(z)\frac{dz}{z} & \varphi(z) = -\frac{1}{2\pi i}\displaystyle\int_{c_*-i\infty}^{c_*+i\infty} e^{z\zeta}\, d\overset{\smile}{\varphi}(\zeta) \\
\text{lower} & \overset{\vee}{\varphi}(\zeta) = \displaystyle\int_c^{+\infty} e^{-\zeta z}\varphi(z)dz & \varphi(z) = \dfrac{1}{2\pi i}\displaystyle\int_{c_*-i\infty}^{c_*+i\infty} e^{z\zeta}\, \overset{\vee}{\varphi}(\zeta)\, d\zeta.
\end{array}$$

Interpretation: Let us assume for definiteness that $0 < \zeta \ll 1 \ll z$. In the Borel-Laplace integrals from z to ζ the constant $c \gg 1$ has to be taken large enough to leave all singularities of the integrand to its left, *i.e.* in $\Re(z) < c$. In the Borel integral from $\overset{\smile}{\varphi}(\zeta)$ or $\overset{\vee}{\varphi}(\zeta)$ to $\varphi(z)$, on the other hand, any positive c_*, large or small, will do, but the integrand $\overset{\smile}{\varphi}(\zeta)$ or $\overset{\vee}{\varphi}(\zeta)$ must be suitably chosen in its equivalence class to ensure integrability (which is always possible, for any given c_* or even for all c_* at once).

Monomials in all four models
The following table covers not only the monomials $J_\sigma(z) := z^{-\sigma}$ but also[18] the whole range of binomials $J_{\sigma,n}(z) := z^{-\sigma}\log^n(z)$ with $\sigma \in \mathbb{C}, n \in \mathbb{N}$.

$$
\begin{array}{lllll}
 & & \text{minor} & & \text{major} \\
\sigma \notin \mathbb{Z} & \text{upper} & \widehat{J}_\sigma(\zeta) = \zeta^\sigma/\Gamma(1+\sigma), & & \breve{J}_\sigma(\zeta) = \zeta^\sigma\Gamma(-\sigma) \\
 & \nearrow\swarrow & & & \\
J_\sigma(z) = z^{-\sigma} & & \downarrow \partial_\zeta & & \downarrow -\partial_\zeta \\
 & \searrow\nwarrow & & & \\
 & \text{lower} & \hat{J}_\sigma(\zeta) = \zeta^{\sigma-1}/\Gamma(\sigma) & , & \overset{\vee}{J}_\sigma(\zeta) = \zeta^{\sigma-1}\Gamma(1-\sigma) \\
\cdots & \cdots & \cdots & & \cdots \\
s \in \mathbb{N}^+ & \text{upper} & \widehat{J}_s(\zeta) = \frac{1}{s!}\,\zeta^s & , & \breve{J}_s(\zeta) = (-\zeta)^s \log\left(\frac{1}{\zeta}\right) \\
 & \nearrow\swarrow & & & \\
J_s(z) = z^{-s} & & \downarrow \partial_\zeta & & \downarrow -\partial_\zeta \\
 & \searrow\nwarrow & & & \\
 & \text{lower} & \hat{J}_s(\zeta) = \frac{1}{(s-1)!}\,\zeta^{s-1} & , & \overset{\vee}{J}_s(\zeta) = (-\zeta)^{s-1}\log\left(\frac{1}{\zeta}\right) \\
\cdots & \cdots & \cdots & & \cdots \\
 & \text{upper} & \widehat{J}_0(\zeta) = 1 & , & \breve{J}_s(\zeta) = \log\left(\frac{1}{\zeta}\right) \\
 & \nearrow\swarrow & & & \\
J_0(z) = 1 & & \downarrow \partial_\zeta & & \downarrow -\partial_\zeta \\
 & \searrow\nwarrow & & & \\
 & \text{lower} & \hat{J}_0(\zeta) = 0 & , & \overset{\vee}{J}_s(\zeta) = \frac{1}{\zeta} \\
\cdots & \cdots & \cdots & & \cdots \\
s \in \mathbb{N}^+ & \text{upper} & \widehat{J}_{-s}(\zeta) = 0 & , & \breve{J}_{-s}(\zeta) = (s-1)!\,\zeta^{-s} \\
 & \nearrow\swarrow & & & \\
J_{-s}(z) = z^s & & \downarrow \partial_\zeta & & \downarrow -\partial_\zeta \\
 & \searrow\nwarrow & & & \\
 & \text{lower} & \hat{J}_{-s}(\zeta) = 0 & , & \overset{\vee}{J}_{-s}(\zeta) = s!\,\zeta^{-s-1}
\end{array}
$$

The pros and cons of the upper/lower choices
Advantages of the lower choice:

(i) $\{\underline{\mathcal{B}}, \underline{\mathcal{L}}, \underline{*}\}$ are more usual/natural choices than $\{\overline{\mathcal{B}}, \overline{\mathcal{L}}, \overline{*}\}$;

[18] After σ-differentiation.

(ii) the operators $(\partial_z + \omega)^{-1}$ and $(e^{\omega\partial_z} - 1)^{-1}$ which constantly occur in the theory of *singular* differential or difference equations and are ultimately responsible for the frequent occurrence, in this theory, of both divergence and resurgence, turn into *minor* multiplication by $(-\zeta + \omega)^{-1}$ or $(e^{-\omega\zeta} - 1)^{-1}$ in the ζ-plane[19], whereas with the upper choice we would be saddled with the more unwieldy operators $\partial_\zeta^{-1}(-\zeta + \omega)^{-1}\partial_\zeta$ and $\partial_\zeta^{-1}(e^{-\omega\zeta} - 1)^{-1}\partial_\zeta$.

Advantages of the upper choice:

(i) the "monomial" formulas for J_σ (*supra*) assume a smoother shape, with the simple sign change $-\sigma \mapsto \sigma$ instead of $-\sigma \mapsto \sigma - 1$;
(ii) upper convolution $\overline{*}$ and pointwise multiplication (both in the ζ-plane) have the same unit element, namely $\widehat{\varphi_0}\ (\zeta) \equiv 1$, which is extremely useful when studying *dimorphy phenomena*[20], *e.g.* the dimorphy of poly- or hyperlogarithms.

In the present investigation, we shall resort to both choices, because:

(i) the lower choice leads to simpler formulas when deriving a function's singularities from its Taylor coefficient asymptotics (see Section 2.3 *infra*);
(ii) the upper choice is the one naturally favoured by the functional transforms (*nir*/*mir* and *nur*/*mur*) that lead to the *inner* and *outer* generators of SP-series (see Section 4-5 and Section 5.3-5 *infra*).

2.2 Alien derivations as a tool for Riemann surface description

Resurgent functions are acted upon by a huge range of exotic derivations, the so-called *alien derivations* Δ_ω, with indices ω ranging through the whole of $\mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$. In other words, $\arg(\omega)$ is defined *exactly* rather than *mod* 2π. Together, these Δ_ω generate a *free Lie algebra* on $\mathbb{C}_\bullet$. Alien derivations, by pull-back, act on all three models. There being no scope for confusion, the same symbols Δ_ω can be used in each model. *Alien derivations*, however, are linear operators which quantitatively measure the *singularities of minors* in the ζ-plane. To interpret or calculate *alien derivatives*, we must therefore go to (either of) the *convolutive models*, which in that sense enjoy an undoubted primacy. However,

[19] Or *major* multiplication by $(\zeta + \omega)^{-1}$ or $(e^{\omega\zeta} - 1)^{-1}$.

[20] *I.e.* the simultaneous stability of certain function rings unter *two* unrelated products, like pointwise *multiplication* and some form or other of *convolution*.

for notational ease, it is often convenient to write down *resurgence equations*[21] in the multplicative models (formal or geometric), the product there being the more familiar multiplication.

For simplicity, in all the following definitions/identities the indices ω are assumed to be on $\mathbb{R}^+ \subset \mathbb{C}_\bullet$. Adaptation to the general case is immediate.

Definition of the operators Δ_ω and $\Delta_\omega^\pm$

multiplicative	convolutive	convolutive
$\Delta_\omega : \quad \varphi \mapsto \varphi_\omega$	$\overset{0}{\varphi} := \{\overset{\frown}{\varphi}, \overset{\smile}{\varphi}\} \mapsto$	$\overset{0}{\varphi}_\omega := \{\overset{\frown}{\varphi}_\omega, \overset{\smile}{\varphi}_\omega\}$
	$\overset{\lozenge}{\varphi} := \{\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}\} \mapsto$	$\overset{\lozenge}{\varphi}_\omega := \{\overset{\wedge}{\varphi}_\omega, \overset{\vee}{\varphi}_\omega\}$

$$\overset{\vee}{\varphi}_\omega(\zeta) := \sum_{\epsilon_1,\dots,\epsilon_{r-1}} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\wedge}{\varphi}^{\binom{\epsilon_1,\dots,\epsilon_{r-1}}{\omega_1,\dots,\omega_{r-1}}}(\omega-\zeta)$$

$$\overset{\wedge}{\varphi}_\omega(\zeta) := \sum_{\epsilon_1,\dots,\epsilon_r} \frac{\epsilon_r}{2\pi i} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\wedge}{\varphi}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}(\omega+\zeta)$$

$$\overset{\vee}{\varphi}_\omega(\zeta) := \sum_{\epsilon_0,\dots,\epsilon_{r-1}} \frac{\epsilon_0}{2\pi i} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\vee}{\varphi}^{\binom{\epsilon_0,\dots,\epsilon_{r-1}}{\underline{\omega}_0,\dots,\underline{\omega}_{r-1}}}(\underline{\omega}+\zeta)$$

$$\overset{\wedge}{\varphi}_\omega(\zeta) := \sum_{\epsilon_0,\dots,\epsilon_r} \frac{\epsilon_0\,\epsilon_r}{2\pi i} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\vee}{\varphi}^{\binom{\epsilon_0,\dots,\epsilon_r}{\underline{\omega}_0,\dots,\underline{\omega}_r}}(\underline{\omega}-\zeta)$$

$$0 =: \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{r-2} < \omega_{r-1} < \omega_r := \omega$$
$$\left(\underline{\omega}_i := -\omega_i,\ \forall i\right).$$

The above relations should first be interpreted locally, *i.e.* for $\zeta/\omega \ll 1$, and then extended globally by analytic continuation in ζ. Here $\overset{\wedge}{\varphi}^{\binom{\epsilon}{\omega}}$ or $\overset{\vee}{\varphi}^{\binom{\epsilon}{\omega}}$ denotes the branch corresponding to the left or right circumvention of each intervening singularity ω_i if ϵ_i is $+$ or $-$, and to a branch weightage λ that doesn't depend on the increments ω_i:

$$\lambda_{\epsilon_1,\dots,\epsilon_{r-1}} := \frac{p!\,q!}{r!} \quad \text{with} \quad p := \sum_{\epsilon_i=+} 1,\ q := \sum_{\epsilon_i=-} 1$$
$$\left(\sum_{\epsilon_i} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \equiv 1\right).$$

[21] *I.e.* any relation $E(\varphi, \Delta_{\omega_1}\varphi, \dots, \Delta_{\omega_n}\varphi) = 0$, linear or not, between a resurgent function φ and one or several of its alien derivatives.

The lateral operators $\Delta_\omega^\epsilon : \varphi \mapsto \varphi_{\omega_\epsilon}$ (with index $\epsilon = \pm$) are defined by the same formulas as above, but with weights $\lambda_{\epsilon_1,\dots,\epsilon_{r-1}}$ replaced by the much more elementary $2\pi i \lambda^\epsilon_{\epsilon_1,\dots,\epsilon_{r-1}}$:

$$\begin{aligned} \lambda^\epsilon_{\epsilon_1,\dots,\epsilon_{r-1}} &:= 1 \ \ \textit{if} \ \ \epsilon_1 = \epsilon_2 = \dots = \epsilon_{r-1} = \epsilon \ \in \pm 1 \\ &:= 0 \quad \text{otherwise.} \end{aligned}$$

Thus, the minor-to-major and minor-to-minor formulas read:

$$\begin{aligned} \overset{\vee}{\varphi}_{\omega_\epsilon}(\zeta) &:= \quad 2\pi i \ \overset{\wedge}{\varphi}^{\left(\substack{\epsilon\ ,\dots,\ \epsilon \\ \omega_1,\dots,\omega_{r-1}}\right)}(\omega - \zeta) \\ \overset{\wedge}{\varphi}_{\omega_\epsilon}(\zeta) &:= \sum_{\epsilon_r} \ \epsilon_r \ \overset{\wedge}{\varphi}^{\left(\substack{\epsilon\ ,\dots,\ \epsilon\ ,\ \epsilon_r \\ \omega_1,\dots,\omega_{r-1},\ \omega_r}\right)}(\omega + \zeta). \end{aligned}$$

This settles the action of alien operators in the *lower* convolutive model. Their action in the *upper* model is exactly the same. Their action in the multiplicative models is defined indirectly, by pull-back from the convolutive models (with the same notation Δ_ω holding for all models).

The operators Δ_ω are derivations but the simpler $\Delta_\omega^\pm$ are not
Indeed, for any two test functions φ_1, φ_2 the identities hold[22] :

$$\begin{aligned} \Delta_\omega(\varphi_1\,\varphi_2) &\equiv (\Delta_\omega\varphi_1)\,\varphi_2 + \varphi_1\,(\Delta_\omega\varphi_2) \\ \Delta_\omega^\pm(\varphi_1\,\varphi_2) &\equiv (\Delta_\omega^\pm\varphi_1)\,\varphi_2 + \varphi_1\,(\Delta_\omega^\pm\varphi_2) + \sum_{\omega_1+\omega_2=\omega}^{\frac{\omega_1}{\omega}>0,\,\frac{\omega_2}{\omega}>0} (\Delta_{\omega_1}^\pm\varphi_1)(\Delta_{\omega_2}^\pm\varphi_2). \end{aligned}$$

Lateral and median singularities
The lateral and median operators are related by the following identities:

$$1 + \sum_{\omega>0} t^\omega\,\Delta_\omega^\pm = \exp\left(\pm 2\pi i \sum_{\omega>0} t^\omega\,\Delta_\omega\right) \tag{2.2}$$

$$2\pi i \sum_{\omega>0} t^\omega\,\Delta_\omega = \pm\log\left(1 + \sum_{\omega>0} t^\omega\,\Delta_\omega^\pm\right). \tag{2.3}$$

Interpretation: we first expand *exp* and *log* the usual way, then equate the contributions of each power t^ω from the left- and right-hand sides. Although the above formulas express each $\Delta_\omega^\pm$ as an infinite sum of (finite)

[22] For simplicity, we write the following identities in the *multiplicative* models. When transposing them to the ζ-plane, where they make more direct sense, multiplication must of course be replaced by convolution.

Δ_ω-products, and *vice versa*, when applied to any given test function φ the infinite sums actually reduce to a finite number of *non-vanishing* summands.[23]

Compact description of Riemann surfaces

Knowing *all* the alien derivatives (of first and higher orders) of a minor $\overset{\wedge}{\varphi}(\zeta)$ or $\overset{\frown}{\varphi}(\zeta)$ enables one to piece together that minor's behaviour on its *entire Riemann surface* $\mathcal{R}$ from the behaviour of its various alien derivatives on their sole *holomorphy stars*, by means of the general formula:

$$\overset{\wedge}{\varphi}(\zeta_\Gamma) \equiv \overset{\wedge}{\varphi}(\zeta) + \sum_{r\geq 1} \sum_{\omega_i \in \mathbb{C}_\bullet} (2\pi i)^r H_\Gamma^{\omega_1,\ldots,\omega_r} t^{\omega_1+\ldots\omega_r} \Delta_{\omega_r} \ldots \Delta_{\omega_1} \overset{\wedge}{\varphi}(\zeta). \quad (2.4)$$

Here, ζ_Γ denotes any chosen point on $\mathcal{R}$, reached from 0 by following a broken line Γ in the ζ-plane. Both sums $\sum_r$ and $\sum_{\omega_i}$ are finite.[24] The coefficients $H_\Gamma^\bullet$ are in $\mathbb{Z}$. Unlike in (2.2), (2.3), t^ω in (2.4) should no longer be viewed as the symbolic power of a free variable t, but as an shift operator acting on functions of ζ and changing ζ into $\zeta + \omega$. [25]

To sum up:

(i) alien derivations "uniformise" everything;
(ii) a full knowledge of a minor's alien derivatives (given for example by a *complete* system of resurgence equations) implies a full knowlege of that minor's Riemann surface.

Strong versus weak resurgence

"Proper" resurgence equations are relations of the form:

$$E(\varphi, \Delta_\omega\varphi) \equiv 0 \qquad \text{or} \qquad E(\varphi, \Delta_{\omega_1}\varphi, \ldots, \Delta_{\omega_n}\varphi) \equiv 0 \qquad (2.5)$$

with expressions E that are typically non-linear (at least in φ) and that may involve arbitrary scalar- or function-valued coefficients. Such equations express unexpected *self-reproduction* properties – that is to say, non-trivial relations between the minor (as a germ at $\zeta = 0$) and its various singularities. Moreover, when the resurgent function φ, in the multiplicative model, happens to be the formal solution of some equation or system

[23] Due to the minors having only isolated singularities.

[24] That is to say, when applied to any given resurgent function, they carry only finitely many non-vanishing terms.

[25] With ζ close to 0 and suitably positioned, to ensure that $\zeta + \omega$ be in the holomorphy star of the test function.

$S(\varphi) = 0$ (think for example of a singular differential, or difference, or functional, equation), the resurgence of φ as well as the exact shape of its resurgence equations (2.5), can usually be derived almost without analysis, merely by letting each Δ_ω act on $S(\varphi)$ in accordance with certain formal rules. Put another way: we can deduce deep *analytic* facts from purely *formal-algebraic* manipulations. What we have here is *full-fledged resurgence* – resurgence at its best and most useful.

But two types of situations may arise which lead to *watered-down* forms of resurgence.

One is the case when, due to severe constraints built into the resurgence-generating problem, the coefficients inside E are no longer free to vary continuously, but must assume discrete, usually entire values: we then speak of *rigid resurgence*.

Another is the case when the expressions E are linear or affine functions of their arguments φ and $\Delta_{\omega_i}\varphi$. The self-reproduction aspect, to which resurgence owes its name, then completely disappears, and makes way for a simple *exchange* or *ping-pong* between singularities (in the linear case) with possible "annihilations" (in the affine case).

Both restrictions entail a severe impoverishment of the resurgence phenomenon. As it happens, and as we propose to show in this paper, SP-series combine these two restrictions: they lead to fairly degenerate resurgence patterns that are both *rigid* and *affine*. Furthermore, as a rule, SP-series verify no useful equation or system $S(\varphi) = 0$ that might give us a clue as to their resurgence properties. In cases such as this, the resurgence apparatus (alien derivations etc.) ceases to be a vehicle for *proving things* and retains only its (non-negligible !) *notational value* (as a device for describing Riemann surfaces etc.) while the onus of proving the hard analytic facts falls on altogether different tools, like *Taylor coefficient asymptotics*[26] and the *nir/mir*-transforms.[27]

The pros and cons of the $2\pi i$ factor

On balance, we gain more than we lose by inserting the $2\pi i$ factor into the above definitions of alien derivations. True, by removing it there we would also eliminate it from the identities relating minors to majors (see Section 2.1), but the factor would sneak back into the J_σ-identities *supra*, thus spoiling the whole set of "monomial" formulas. Worse still, real-indexed derivations Δ_ω acting on real-analytic derivands φ would no

[26] See Section 2.3.

[27] See Section 4.4, Section 4.5.

longer produce real-analytic derivatives $\Delta_\omega\varphi$ – which would be particularly damaging in "all-real" settings, *e.g.* when dealing with chirality 1 knots like 4_1 (see Section 9 *infra*).

2.3 Retrieving the resurgence of a series from the resurgence of its Taylor coefficients

SP-series are one of those rare instances where there is no shortcut for calculating the singularities: we have no option but to deduce them from a close examination of the asymptotics of the Taylor coefficients.[28]

The better to respect the symmetry between our series φ and its Taylor coefficients J, we shall view them both as resurgent functions of the variables z respectively n in the multiplicative models and ζ respectively ν in the (lower) convolutive models. The aim then is to understand the correspondence between the triplets:

$$\{\tilde{\varphi}(z), \overset{\diamond}{\varphi}(\zeta), \varphi(z)\} \longleftrightarrow \{\tilde{J}(n), \overset{\diamond}{J}(\nu), J(n)\} \tag{2.6}$$

and the alien derivatives attached to them.

Retrieving closest singularities

Let us start with the simplest case, when $\overset{\wedge}{\varphi}$ has a single singularity on the boundary of its disk of convergence, say at ζ_0. We can of course assume ζ_0 to be real positive.

$$\tilde{\varphi}(z) = \sum_{0\leq n}(n+1)!\, J(n)\, z^{-n-1} \quad (\text{divergent}^t) \tag{2.7}$$

$$\overset{\mathcal{B}}{\mapsto}\overset{\wedge}{\varphi}(\zeta) = \sum_{0\leq n} J(n)\, \zeta^n \quad (\text{convergent}^t \text{ on } |\zeta| < \zeta_0). \tag{2.8}$$

In order to deduce the closest singularity of $\overset{\wedge}{\varphi}$ from the closest singularity of $\hat{J}$, we first express $J(n)$ as a Cauchy integral on a circle $|\zeta| = |\zeta_0| - \epsilon$. We then deform that circle to a contour $\mathcal{C}$ which coincides with the larger circle $|\zeta| = |\zeta_0| + \epsilon$ except for a slit Γ around the interval $[\zeta_0, \zeta_0 + \epsilon]$ to avoid crossing the axis $[\zeta_0, \infty]$. Then we retain only Γ, thereby neglecting a contribution exponentially small in n. Lastly, we transform

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[28] The present section is based on a private communication (1992) by J.E. to Prof. G.K. Immink. An independent, more detailed treatment was later given by O. Costin in [2].

Γ into Γ_* (respectively $\underline{\Gamma_*} = -\Gamma_*$) under the change $\zeta = \zeta_0 e^\nu$.

$$J(n) = \frac{1}{2\pi i} \oint \overset{\wedge}{\varphi} (\zeta)\, \zeta^{-n-1}\, d\zeta \tag{2.9}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \overset{\wedge}{\varphi} (\zeta)\, \zeta^{-n-1}\, d\zeta \qquad \text{(contour deformation)}$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \overset{\wedge}{\varphi} (\zeta)\, \zeta^{-n-1}\, d\zeta + o(\zeta_0^{-n}) \quad \text{(contour deformation)} \tag{2.10}$$

$$= \frac{e^{-n\nu_0}}{2\pi i} \int_{\Gamma_*} \overset{\wedge}{\varphi} (\zeta_0 e^\nu)\, e^{-n\nu}\, d\nu + o(e^{-n\nu_0})$$

$$\text{(setting } \zeta := \zeta_0\, e^\nu = e^{\nu_0+\nu})$$

$$= \frac{e^{-n\nu_0}}{2\pi i} \int_{\Gamma_*} \overset{\vee}{\varphi}_{\zeta_0} (\zeta_0 - \zeta_0 e^\nu)\, e^{-n\nu}\, d\nu + o(e^{-n\nu_0}) \quad \text{(always)} \tag{2.11}$$

$$= \frac{e^{-n\nu_0}}{2\pi i} \int_{\underline{\Gamma_*}} \overset{\vee}{\varphi}_{\zeta_0} (\zeta_0 - \zeta_0 e^{-\nu})\, e^{n\nu}\, d\nu + o(e^{-n\nu_0}) \quad \text{(always)} \tag{2.12}$$

$$= e^{-n\nu_0} \int_0^c \overset{\wedge}{\varphi}_{\zeta_0} (\zeta_0 e^\nu - \zeta_0)\, e^{-n\nu} d\nu + o(e^{-n\nu_0}) \quad (\text{if } \overset{\Diamond}{\varphi}_{\zeta_0} \text{ integrable}). \tag{2.13}$$

Therefore

$$J(n) \equiv e^{-n\nu_0}\, J_{\nu_0}(n) + o(e^{-n\nu_0}) \qquad \text{(with } \nu_0 := \log(\zeta_0)) \tag{2.14}$$

where J_{ν_0} denotes the (lower, and if need be, truncated) Borel transform of a resurgent function $\overset{\Diamond}{J}_{\nu_0}$ linked to $\overset{\Diamond}{\varphi}_{\zeta_0} := \Delta_{\zeta_0} \overset{\wedge}{\varphi}$ by:

$$\overset{\Diamond}{J}_{\nu_0} (\nu) = \{\hat{J}_{\nu_0} (\nu), \overset{\vee}{J}_{\nu_0} (\nu)\} \qquad \overset{\Diamond}{\varphi}_{\zeta_0} (\nu) = \{\overset{\wedge}{\varphi}_{\zeta_0} (\nu), \overset{\vee}{\varphi}_{\zeta_0} (\zeta)\}$$

$$\hat{J}_{\nu_0} (\nu) = \overset{\wedge}{\varphi}_{\zeta_0} (\zeta_0\, e^\nu - \zeta_0\,) \qquad \overset{\wedge}{\varphi}_{\zeta_0} (\zeta) = \hat{J}_{\nu_0} \left(\log \left(1 + \frac{\zeta}{\zeta_0} \right) \right) \quad \text{(minors)}$$

$$\overset{\vee}{J}_{\nu_0} (\nu) = \overset{\vee}{\varphi}_{\zeta_0} (\zeta_0 - \zeta_0\, e^{-\nu}) \qquad \overset{\vee}{\varphi}_{\zeta_0} (\zeta) = \overset{\vee}{J}_{\nu_0} \left(-\log \left(1 - \frac{\zeta}{\zeta_0} \right) \right) \quad \text{(majors).}$$

Retrieving distant singularities

The procedure actually extends to farther-lying singularities. In fact, if $\hat{J}$ is endlessly continuable, so is $\overset{\wedge}{\varphi}$, and the former's resurgence pattern neatly translates into the latter's, under a set of linear but non-trivial formulas. Here, however, we shall only require knowledge of those singularities of $\overset{\wedge}{\varphi}$ which lie on its (0-centered, closed) star of holomorphy. All the other singularities will follow under repeated alien differentiation.

3 The ingress factor

We must first describe the asymptotics of the "product" part (for $m = n$) of our "sum-product" coefficients. This involves a trifactorisation:

$$\prod_{0\leq k\leq n} F\left(\frac{k}{n}\right) =: P_F(n) \sim \tilde{Ig}_F(n)\, e^{-\nu_* n}\, \tilde{Eg}_F(n) \qquad \text{with} \quad \nu_* = \int_0^1 f(x)dx \tag{3.1}$$

with

(i) an *ingress* factor $\tilde{Ig}_F$ resummable to Ig_F and purely local at $x = 0$;
(ii) an *exponential* factor $e^{-\nu_* n}$, global on $[0, 1]$;
(iii) an *egress* factor $\tilde{Eg}_F$ resummable to Eg_F and purely local at $x = 1$.

The non-trivial factors (ingress/egress) may be divergent-resurgent (hence the tilda) but, at least for holomorphic data F, they always remain fairly elementary. They often vanish (when F is *even* at 0 or 1) and, even when divergent, they can always be resummed in a canonical way. Lastly, as already hinted, it will prove technically convenient to factor out the first of these (ingress), thereby replacing the original SP-series $j(\zeta)$ by its 'cleansed' and more regular version $j^{\#}(\zeta)$.

3.1 Bernoulli numbers and polynomials

For future use, let us collect a few formulas about two convenient variants of the classical Bernoulli numbers B_k and Bernoulli polynomials $B_k(t)$.

The Bernoulli numbers and polynomials

$$\mathfrak{b}_k := \frac{\mathfrak{b}_k^*(0)}{k!} = \frac{B_{k+1}(1)}{(k+1)!} \qquad (k \in -1 + \mathbb{N}) \tag{3.2}$$

$$\mathfrak{b}(\tau) := \frac{e^\tau}{e^\tau - 1} = \sum_{k\geq -1} \mathfrak{b}_k\, \tau^k = \tau^{-1} + \frac{1}{2} + \frac{1}{12}\tau - \frac{1}{720}\tau^3 \ldots \tag{3.3}$$

$$\mathfrak{b}_k^*(\tau) := \mathfrak{b}(\partial_\tau)\, \tau^k \qquad (k \in \mathbb{C},\ k \neq -1) \tag{3.4}$$

$$\mathfrak{b}^{**}(\tau, \zeta) := \sum_{k\geq 0} \mathfrak{b}_k^*(\tau)\, \frac{\zeta^k}{k!} = \mathfrak{b}(\partial_\tau)\, e^{\tau\zeta} = \frac{e^{\tau\zeta} e^\zeta}{e^\zeta - 1} - \frac{1}{\zeta}. \tag{3.5}$$

For $k \in \mathbb{N}$, we have $\mathfrak{b}_k = \frac{B_{k+1}}{(1+k)!}$ for the scalars, and the series $\mathfrak{b}_k^*(\tau)$ essentially coincide with the Bernoulli polynomials. For all other values of k, the scalars $\mathfrak{b}_k$ are no longer defined and the $\mathfrak{b}_k^*(\tau)$ become divergent series in decreasing powers of τ.

$$\mathfrak{b}_k^*(\tau) := \sum_{s=-1}^{k} \mathfrak{b}_s\, \tau^{k-s} \frac{k!}{(k-s)!} = \frac{B_{k+1}(\tau+1)}{k+1} \qquad (\text{if } k \in \mathbb{N}) \tag{3.6}$$

$$\mathfrak{b}_k^*(\tau) := \sum_{s=-1}^{+\infty} \mathfrak{b}_s\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} \qquad (\text{if } k \in \mathbb{C} - \mathbb{Z}) \tag{3.7}$$

$$\mathfrak{b}_k^*(\tau) := \frac{\tau^{k+1}}{k+1} + \sum_{s\geq 0} (-1)^s \mathfrak{b}_s\, \tau^{k-s} \frac{(s-k-1)!}{(-k-1)!} \qquad (\text{if } k \in -2 - \mathbb{N}). \tag{3.8}$$

The Euler-Bernoulli numbers and polynomials

$$\beta_k := \frac{\beta_k^*(0)}{k!} = \frac{B_{k+1}\left(\frac{1}{2}\right)}{(k+1)!} \qquad (k \in -1 + \mathbb{N}) \tag{3.9}$$

$$\beta(\tau) := \frac{1}{e^{\tau/2} - e^{-\tau/2}} = \sum_{k\geq -1} \beta_k\, \tau^{-k} = \tau^{-1} - \frac{1}{24}\tau + \frac{7}{5760}\tau^3 - \dots \tag{3.10}$$

$$\beta_k^*(\tau) := \beta(\partial_\tau)\, \tau^k \qquad (k \in \mathbb{C},\ k \neq -1) \tag{3.11}$$

$$\beta^{**}(\tau,\zeta) := \sum_{k\geq 0} \beta_k^*(\tau) \frac{\zeta^k}{k!} = \beta(\partial_\tau)\, e^{\tau\zeta} = \frac{e^{\tau\zeta}}{e^{\zeta/2} - e^{-\zeta/2}} - \frac{1}{\zeta}. \tag{3.12}$$

For $k \in \mathbb{N}$, the $\beta_k^*(\tau)$ essentially coincide with the Euler-Bernoulli polynomials. For all other values of k, they are divergent series in decreasing powers of τ.

$$\beta_k^*(\tau) := \sum_{s=-1}^{k} \beta_s \tau^{k-s} \frac{k!}{(k-s)!} = \frac{B_{k+1}\left(\tau + \frac{1}{2}\right)}{k+1} \qquad (\text{if } k \in \mathbb{N}) \tag{3.13}$$

$$\beta_k^*(\tau) := \sum_{s=-1}^{+\infty} \beta_s\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} \qquad (\text{if } k \in \mathbb{C} - \mathbb{Z}) \tag{3.14}$$

$$\beta_k^*(\tau) := \frac{\tau^{k+1}}{k+1} + \sum_{s\geq 0} (-1)^s \beta_s\, \tau^{k-s} \frac{(s-k-1)!}{(-k-1)!} \qquad (\text{if } k \in -2 - \mathbb{N}). \tag{3.15}$$

For all $k \in \mathbb{N}$ we have the parity relations $\beta_{2k}^* = 0$, $\beta_k^*(-\tau) \equiv (-1)^{k+1} \beta_k^*(\tau)$.

The Euler-MacLaurin formula
We shall make constant use of the basic identities ($\forall s \in \mathbb{N}$):

$$\sum_{1\leq k\leq m} k^s \equiv \mathfrak{b}_s^*(m) - \mathfrak{b}_s^*(0) \equiv \beta_s^*\left(m+\frac{1}{2}\right) - \beta_s^*\left(\frac{1}{2}\right)$$
$$\equiv \frac{B_{s+1}(m+1) - B_{s+1}(1)}{s+1}$$

and of these variants of the Euler-MacLaurin formula:

$$\sum_{k}^{0\leq\frac{k}{n}\leq\bar{x}} f\left(\frac{k}{n}\right) \sim n\int_0^{\bar{x}} f(x)dx + \frac{f(0)}{2} + \frac{f(\bar{x})}{2} + \sum_{1\leq s \text{ odd}} \frac{\mathfrak{b}_s}{n^s}\big(f^{(s)}(\bar{x}) - f^{(s)}(0)\big) \tag{3.16}$$

$$\sum_{k}^{0\leq\frac{k}{n}\leq\bar{x}} f\left(\frac{k}{n}\right) \sim n\int_0^{\bar{x}} f(x)dx + \frac{f(0)}{2} + \frac{f(\bar{x})}{2} + \sum_{1\leq s \text{ odd}} \frac{\mathfrak{b}_s^*(0)}{n^s}\big(\bar{f}_s - f_s\big) \tag{3.17}$$

where f_s and $\bar{f}_s$ denote the Taylor coefficients of f at 0 and $\bar{x}$.

3.2 Resurgence of the Gamma function

Lemma 3.1 (Exact asymptotics of the Gamma function). *The functions Θ, θ defined on $\{\Re(n) > 0\} \subset \mathbb{C}$ by*

$$\Theta(n) \equiv e^{\theta(n)} := (2\pi)^{-\frac{1}{2}}\Gamma(n+1)\, n^{-n-\frac{1}{2}}\, e^n \quad (\theta(n) \textit{ real if } n \textit{ real}) \tag{3.18}$$

possess resurgent-resummable asymptotic expansions as $\Re(n) \to +\infty$:

$$\Theta(n) = 1 + \sum_{1\leq s} \Theta_s\, n^{-s}; \quad \theta(n) = \sum_{0\leq s} \theta_{1+2s}\, n^{-1-2s} \quad (\textit{odd powers}) \tag{3.19}$$

with explicit lower/upper Borel transforms:

$$\overset{\wedge}{\theta}(\nu) = -\frac{1}{\nu^2} + \frac{1}{2\,\nu}\frac{1}{\tanh(\nu/2)} \tag{3.20}$$

$$\widehat{\theta}(\nu) = +\frac{1}{\nu} + \frac{1}{2}\int_0^{\nu/2} \frac{1}{\tanh(t)}\frac{dt}{t} = \frac{1}{12}\nu - \frac{1}{2160}\nu^3 + \frac{1}{151200}\nu^5 - \frac{1}{8467200}\nu^7 + \frac{1}{431101440}\nu^9 \dots \tag{3.21}$$

This immediately follows from Γ's functional equation. We get successively:

$$\frac{\Theta\left(n+\frac{1}{2}\right)}{\Theta\left(n-\frac{1}{2}\right)} e \left(\frac{n-\frac{1}{2}}{n+\frac{1}{2}}\right)^{n}$$

$$\theta\left(n+\frac{1}{2}\right)-\theta\left(n-\frac{1}{2}\right)=1+n\log\left(n-\frac{1}{2}\right)-n\log\left(n+\frac{1}{2}\right)$$

$$\partial_n \frac{1}{n}\left(\theta\left(n+\frac{1}{2}\right)-\theta\left(n+\frac{1}{2}\right)\right) = -\frac{1}{n^2}+\frac{1}{n-\frac{1}{2}}-\frac{1}{n+\frac{1}{2}}$$

$$-\nu\,\partial_\nu^{-1}\left((e^{-\nu/2}-e^{\nu/2})\,\overset{\wedge}{\theta}\,(\nu)\right) = -\nu+e^{\nu/2}-e^{-\nu/2}$$

$$\overset{\wedge}{\theta}\,(\nu)=(e^{\nu/2}-e^{-\nu/2})^{-1}\partial_\nu \nu^{-1}(e^{\nu/2}-e^{-\nu/2})$$

$$\overset{\wedge}{\theta}\,(\nu) = -\frac{1}{\nu^2}+\frac{1}{2\nu}\frac{1}{\tanh(\nu/2)}.$$

Laplace summation along $\arg(\nu) = 0$ yields the exact values $\theta(n)$ and $\Theta(n)$. The only non-vanishing alien derivatives are (in multiplicative notation):

$$\Delta_\omega\,\tilde{\theta} = \frac{1}{\omega} \qquad \forall\omega\in 2\pi i\mathbb{Z}^* \tag{3.22}$$

$$\Delta_\omega\,\tilde{\Theta} = \frac{1}{\omega}\,\tilde{\Theta} \qquad \forall\omega\in 2\pi i\mathbb{Z}^*. \tag{3.23}$$

Using formula (3.23) and its iterates for crossing the vertical axis in the ν-plane, we can evaluate the quotient of the regular resummations of $\hat{\Theta}(\nu)$ along $\arg(\nu) = 0$ and $\arg(\nu) = \pm\pi$, and the result of course agrees with the complement formula[29]:

$$\frac{1}{\Gamma(n)\,\Gamma(1-n)} = \frac{\sin\pi\,n}{\pi} \qquad \forall n\in\mathbb{C} \tag{3.24}$$

[29] For details, see [9, pages 243-244].

3.3 Monomial/binomial/exponential factors

In view of definition (3.1) and formula (3.16), for a generic input $F := e^{-f}$ with $F(0)$, $F(1) \neq 0, \infty$ we get the asymptotic expansions:

$$\tilde{Ig}_F(n) = \exp\left(-\frac{1}{2}f(0) + \sum_{1\leq s \text{ odd}} \frac{\mathfrak{b}_s}{n^s} f^{(s)}(0)\right) \tag{3.25}$$

$$\tilde{Eg}_F(n) = \exp\left(-\frac{1}{2}f(1) - \sum_{1\leq s \text{ odd}} \frac{\mathfrak{b}_s}{n^s} f^{(s)}(1)\right) \tag{3.26}$$

and the important parity relation[30]:

$$\begin{aligned} &\{F^{\vDash}(x) = 1/F(1-x)\} \\ &\qquad\Longrightarrow \{1 = \tilde{Ig}_F(n)\tilde{Eg}_{F^{\vDash}}(n) = \tilde{Ig}_{F^{\vDash}}(n)\tilde{Eg}_F(n).\} \end{aligned} \tag{3.27}$$

But we are also interested in meromorphic inputs F that may have zeros and poles at 0 or 1. Since the mappings $F \mapsto \tilde{Ig}_F$ and $F \mapsto \tilde{Eg}_F$ are clearly multiplicative and since meromorphic functions F possess convergent Hadamard products:

$$F(x) = c\, x^d\, e^{\sum_{s=1}^{s=\infty} c_s\, x^s} \prod_i \left(\left(1 - \frac{x}{a_i}\right)^{k_i} e^{k_i \sum_{s=1}^{s=K_i} \frac{1}{s}\frac{x^s}{a_i^s}}\right) \qquad (k_i \in \mathbb{Z},\, K_i \in \mathbb{N}) \tag{3.28}$$

we require the exact form of the ingress factors for monomial, binomial and even/odd exponential factors. Here are the results:

$$\begin{aligned}
&F_{\text{mon}}(x) = c && \tilde{Ig}_{\text{mon}}(n) = c^{+\frac{1}{2}} \\
&F_{\text{mon}}(x) = c\, x^d \quad (d \neq 0) && \tilde{Ig}_{\text{mon}}(n) = c^{-\frac{1}{2}}\, (2\pi n)^{\frac{d}{2}} \\
&F_{\text{bin}}(x) = \prod_i (1 - a_i^{-1} x)^{s_i} && \tilde{Ig}_{\text{bin}}(n) = \prod_i \left(\tilde{\Theta}(a_i n)\right)^{s_i} \\
&F_{\text{even}}(x) = \exp\left(-\sum_{s\geq 1} f_{2s}^{\text{even}}\, x^{2s}\right) && \tilde{Ig}_{\text{even}}(n) = 1 \\
&F_{\text{odd}}(x) = \exp\left(-\sum_{s\geq 0} f_{2s+1}^{\text{odd}} x^{2s+1}\right) && \tilde{Ig}_{\text{odd}}(n) = \exp\left(+\sum_{s\geq 0} f_{2s+1}^{\text{odd}} \frac{\mathfrak{b}_{2s+1}^*(0)}{n^{2s+1}}\right)
\end{aligned}$$

[30] In Section 5 it will account for the relation between the two outer generators which are always present in the generic case (*i.e.* when $F(0)$ and $F(1) \neq 0, \infty$).

Monomial factors

The discontinuity between the first two expressions of $Ig_{\mathrm{mon}}(n)$ stems from the fact that for $d=0$ the product in (1.2) start from $k=0$ as usual, whereas for $d\neq 0$ it has to start from $k=1$. The case $d=0$ is trivial, and the case $d\neq 0$ by multiplicativity reduces to the case $d=1$. To calculate the corresponding ingress factor, we may specialise the identity

$$\prod_{1\leq k\leq n} F\left(\frac{k}{n}\right) \sim \tilde{Ig}_F(n)\; e^{-\nu_* n}\; \tilde{Eg}_F(n) \tag{3.29}$$

to convenient test functions. Here are the two simplest choices:

test function $F_1(x)=x$	test function $F_2(x)=\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)$	
$\prod_{k=1}^{k=n} F_1\left(\frac{k}{n}\right)=\frac{n!}{n^n}$	$\prod_{k=1}^{k=n} F_2\left(\frac{k}{n}\right)=2\pi^{-n}n^{1/2}$	by elem. trigon.
$\tilde{Ig}_{F_1}(n)=\text{unknown}$	$\tilde{Ig}_{F_2}(n)=\tilde{Ig}_{F_1}(n)$	by parity of $\frac{F_2(x)}{F_1(x)}$
$\nu_*=1$	$\nu_*=-\log\pi$	
$\tilde{Eg}_{F_1}(n)=\tilde{\Theta}(n)$	$\tilde{Eg}_{F_2}(n)=1$	by parity of $F_2(1+x)$.

With the choice F_2, all we have to do is plug the data in the second column into (3.29) and we immediately get $\tilde{Ig}_{F_2}(n)=(2\pi\, n)^{1/2}$ but before that we have to check the first line's elementary trigonometric identity. With the choice F_1, on the other hand, we need to check that the egress factor does indeed coincide with Θ. This readily follows from:

$$F_1(1+x)=\exp\left(\sum_{1\leq s}(-1)^s\frac{x^s}{s}\right) \Longrightarrow \tilde{Eg}_{F_1}(n)=\exp(\tilde{eg}_{F_1}(n)) \quad \text{with}$$

$$\tilde{eg}_{F_1}(n)=\sum_{1\leq s \text{ odd}}(-1)^{s-1}\frac{n^{-s}}{s}\mathfrak{b}_s^*(0)=\sum_{1\leq s \text{ odd}} n^{-s}(s-1)!\mathfrak{b}_s \Longrightarrow$$

$$\hat{e}\,g_{F_1}(\nu)=\sum_{1\leq s}\nu^{s-1}\mathfrak{b}_s=\frac{1}{\nu}\left(\frac{e^\nu}{e^\nu-1}-\frac{1}{\nu}-\frac{1}{2}\right)$$
$$=-\frac{1}{\nu^2}+\frac{1}{2\,\nu}\,\frac{1}{\tanh(\nu/2)}=\hat{\theta}\,(\nu).$$

We then plug everything into (3.29) and use formula (3.18) of Section 3.2 to eliminate both $n!/n^n$ and $\Theta(n)$.

Binomial factors

By multiplicativity and homogeneity, it is enough to check the idendity $\tilde{Ig}_{F_3}(n) = \tilde{\Theta}(n)$ for the test function $F_3(x) = 1 - x$. But since $F_3 = 1/F_1^{\vDash}$ with the notations of the preceding para, the parity relations yield:

$$\tilde{Ig}_{F_3}(n) = 1/\tilde{Eg}_{1/F_1^{\vDash}}(n) = \tilde{Eg}_{F_1^{\vDash}}(n) = \tilde{\Theta}(n) \qquad \text{(see above)}$$

which is precisely the required identity. We alse notice that:

$$F(x) = \left(1 - \frac{x}{a}\right)\left(1 + \frac{x}{a}\right) \quad \Longrightarrow \quad \tilde{Ig}_F(n) = \tilde{\Theta}(an)\tilde{\Theta}(-an) \equiv 1$$

which agrees with the trivialness of the ingress factor for an *even* input F.

Exponential factors

For them, the expression of the ingress/egress factors directly follows from (3.17). Moreover, since the exponentials occurring in the Hadamard product (3.28) carry only polynomials or entire functions, the corresponding ingress/egress factors are actually convergent.

3.4 Resummability of the total ingress factor

As announced, we shall have to change our SP-series $j(\zeta) = \sum J(n)\,\zeta^n$ into $j^{\#}(\zeta) = \sum J^{\#}(n)\,\zeta^n$, which involves dividing the coefficients $J(n)$, not by the asymptotic series $\tilde{Ig}_F(n)$, but by its exact resummation $Ig_F(n)$. Luckily, this presents no difficulty for meromorphic[31] inputs F. Indeed, the contributions to $\tilde{Ig}_F$ of the isolated factors in (3.28) are separetely resummable:

- for the monomial factors F_{mon} this is trivial;
- for the binomial factors F_{bin} this follows from $\tilde{\Theta}$'s resummability (see Section 3.2);
- for the exponential factors F_{exp} this follows from $\log F_{exp}$ being either polynomial or entire.

As for the global $\tilde{Ig}_F$, one easily checks that the Hadamard product (3.28), rewritten as

$$F = F_{\mathrm{mon}}\ F_{\mathrm{even}}\ F_{\mathrm{odd}} \prod_i F_{\mathrm{bin},i} \tag{3.30}$$

translates into a product of resurgent functions:

$$\tilde{Ig}_F = \tilde{Ig}_{F_{\mathrm{mon}}}\ \tilde{Ig}_{F_{\mathrm{even}}}\ \tilde{Ig}_{F_{\mathrm{odd}}} \prod_i \tilde{Ig}_{F_{\mathrm{bin},i}} \tag{3.31}$$

[31] For simplicity, let us assume that F has no purely imaginary poles or zeros.

which converges in all three models (formal, convolutive, geometric – respective to the corresponding topology) to a limit that doesn't depend on the actual Hadamard decomposition chosen in (3.28), *i.e.* on the actual choice of the truncation-defining integers K_i.

3.5 Parity relations

Starting from the elementary parity relations for the Bernoulli numbers and polynomials:

$$\begin{array}{llll} \mathfrak{b}_{2s} = 0 & (s \geq 1); & \beta_{2s} = 0 & (s \geq 0) \\ \mathfrak{b}_s(\tau) \equiv (-1)^{s+1}\, \mathfrak{b}_s(-\tau-1); & & \beta_s(\tau) \equiv (-1)^{s+1}\, \beta(-\tau) & (\forall s \geq 0) \end{array}$$

and setting

$$\begin{aligned} F^{\vDash}(x) &:= 1/F(1-x) \\ P_F(n) &:= \prod_{m=0}^{m=n} F\left(\frac{m}{n}\right) \\ P_F^{\#}(n) &:= \frac{P_F(n)}{Ig_F(n)\; Eg_F(n)} = (\omega_F)^n \qquad \text{with } \omega_F = \exp\left(-\int_0^1 f(x)dx\right) \end{aligned}$$

we easily check that:

$$\begin{array}{lcl} \tilde{Ig}_{F}(n)\; \tilde{Eg}_{F^{\vDash}}(n) = 1 & \text{and} & \tilde{Ig}_{F^{\vDash}}(n)\; \tilde{Eg}_{F}(n) = 1 \\ J_{F^{\vDash}}(n) = J_F(n)/P_F(n) & \text{and} & J^{\#}_{F^{\vDash}}(n) = J^{\#}_F(n)/P^{\#}_F(n) \\ j_{F^{\vDash}}(\zeta) \neq j_F(\zeta/\omega_F) & \text{but} & j^{\#}_{F^{\vDash}}(\zeta) = j^{\#}_F(\zeta/\omega_F) \end{array}$$

4 Inner generators

4.1 Some heuristics

Consider a simple, yet typical case. Assume the driving function f to be *entire* (or even think of it as *polynomial*, for simplicity), steadily increasing on the real interval $[0, 1]$, with a unique zero at $\bar{x} \in]0, 1[$ on that interval, and no other zeros, real or complex, inside the disk $\{|x| \leq |\bar{x}|\}$:

$$0 < \bar{x} < 1,\;\; f(0) < 0,\;\; f(\bar{x}) = 0,\;\; f(1) > 0,\;\; f'(x) > 0\; \forall x \in [0, 1]. \quad (4.1)$$

As a consequence, the primitive $f^*(x) := \int_0^x f(x')dx'$ will display a unique minimum at $\bar{x}$ and, for any given large n, the products $\prod_{k=0}^{k=m} F(k/n) = \exp(-\sum_{k=0}^{k=m} f(k/n))$ will be maximal for $m \sim n\bar{x}$. It is natural, therefore, to split the Taylor coefficients $J(n)$ of our sum-product series (1.2) into a *global* but fairly elementary factor $J_{1,2,3}(n)$, which subsumes all the *pre-critical* terms $F(k/n)$, and a purely *local* but analytically more challenging factor $J_4(n)$, which accounts for the contribution of all *near-critical* terms $F(k/n)$. Here are the definitions:

$$J(n) = J_{1,2,3}(n)\ J_4(n) \tag{4.2}$$

$$J(n) := \sum_{0\le m\le n} \prod_{0\le k\le m} F\left(\frac{k}{n}\right) \tag{4.3}$$

$$J_{1,2,3}(n) := \prod_{0\le k\le \bar m} F\left(\frac{k}{n}\right) \qquad \text{with} \qquad \bar m := \mathrm{ent}(n\,\bar x) \tag{4.4}$$

$$J_4(n) := \sum_{0\le m\le n} \left(\prod_{0\le k\le m} F\left(\frac{k}{n}\right) \Big/ \prod_{0\le k\le \bar m} F\left(\frac{k}{n}\right)\right) \tag{4.5}$$

$$= \left\{ \begin{array}{c} \cdots + \frac{1}{F}(\frac{\bar m-2}{n})\,\frac{1}{F}(\frac{\bar m-1}{n})\,\frac{1}{F}(\frac{\bar m}{n}) + \frac{1}{F}(\frac{\bar m-1}{n})\,\frac{1}{F}(\frac{\bar m}{n}) \\ +\frac{1}{F}(\frac{\bar m}{n}) + 1 + F(\frac{\bar m+1}{n}) \\ +F(\frac{\bar m+1}{n})\,F(\frac{\bar m+2}{n}) + F(\frac{\bar m+1}{n})\,F(\frac{\bar m+2}{n})\,F(\frac{\bar m+3}{n}) + \ldots \end{array} \right\}. \tag{4.6}$$

The asymptotics of the global factor $J_{1,2,3}(n)$ as $n \to \infty$ easily results from the variant (3.17) of the Euler-MacLaurin formula and $J_{1,2,3}(n)$ splits into three subfactors:

(i) a factor $J_1(n)$, local at $x = 0$, which is none other than the ingress factor $Ig_F(n)$ studied at length in Section 3;
(ii) an elementary factor $J_2(n)$, which reduces to an exponential and carries no divergence;
(iii) a factor $J_3(n)$, local at $x = \bar x$ and analogous to the "egress factor" of Section 3, but with base point $\bar x$ instead of 1.

That leaves the really sensitive factor $J_4(n)$, which like $J_3(n)$ is local at $x = \bar x$, but far more complex. In view of its expression as the discrete sum (4.6), we should expect its asymptotics to be described by a Laurent series $\sum_{k\ge 0} C_{k/2}\, n^{-k/2}$ involving both integral and semi-integral powers of $1/n$. That turns out to be the case indeed, but we shall see that there is a way of jettisoning the integral powers and retaining only the semi-integral ones, *i.e.* $\sum_{k\ge 0} C_{k+1/2}\, n^{-k-1/2}$. To do this, we must perform a little sleight-of-hand and attach the egress factor J_3 to J_4 so as to produce the joint factor $J_{3,4}$. In fact, as we shall see, the gains that accrue from merging J_3 and J_4 go way beyond the elimination of integral powers.

But rather than rushing ahead, let us describe our four factors $J_i(n)$ and their asymptotic expansions $\tilde{J}_i(n)$:

$$J(n) := J_{1,2,3}(n)\, J_4(n) = J_1(n)\, J_2(n)\, J_3(n)\, J_4(n) = J_{1,2}(n)\, J_{3,4}(n)$$

$$J_{1,2,3}(n) := \prod_{0\le k\le \bar{m}} F\left(\frac{k}{n}\right) = \exp\left(-\sum_{0\le k\le \bar{m}} f\left(\frac{k}{n}\right)\right) \quad \text{with } \bar{m} := \mathrm{ent}(n\bar{x})$$

$$\tilde{J}_1(n) := \exp\left(-\frac{1}{2}\, f_0 + \sum_{1\le s} \frac{\mathfrak{b}_s^*(0)\, f_s}{n^s}\right) \quad \text{with } f_s := \frac{f^{(s)}(0)}{s!} \tag{4.7}$$

$$\tilde{J}_2(n) := \exp(-n\, \bar{\nu}) \qquad\qquad \text{with } \bar{\nu} := \int_0^{\bar{x}} f(x)\, dx \tag{4.8}$$

$$\tilde{J}_3(n) := \exp\left(-\frac{1}{2}\, \bar{f}_0 - \sum_{1\le s} \frac{\mathfrak{b}_s^*(0)\, \bar{f}_s}{n^s}\right) \quad \text{with } \bar{f}_s := \frac{f^{(s)}(\bar{x})}{s!} \tag{4.9}$$

$$\begin{aligned}\tilde{J}_4(n) := &\sum_{0\le \bar{m}\le \bar{x}\, n} \exp\left(\sum_{0\le k\le \bar{m}} f\left(\bar{x} - \frac{k}{n}\right)\right)\\ &+ \sum_{0\le \bar{m}\le (1-\bar{x})\, n} \exp\left(-\sum_{1\le k\le \bar{m}} f\left(\bar{x} + \frac{k}{n}\right)\right).\end{aligned}$$

In the last identity, the first exponential inside the second sum, namely $\exp(\sum_{1\le k\le 0}(\ldots))$, should of course be taken as $\exp(0) = 1$. Let us now simplify $\tilde{J}_3$ by using the fact that $\bar{f}_0 = f(\bar{x}) = 0$ and let us replace in $\tilde{J}_4$ the finite $\bar{m}$-summation (up to $\bar{x}\, n$ or $(1-\bar{x})\, n$) by an infinite m-summation, up to $+\infty$, which won't change the *asymptotics*[32] in n:

$$\tilde{J}_3(n) := \exp\left(-\sum_{1\le s} \frac{\mathfrak{b}_s^*(0)}{n^s}\, \bar{f}_s\right) \tag{4.10}$$

$$\tilde{J}_4(n) := 2 + \sum_{\substack{1\le m\\ \epsilon=\pm 1}} \exp\left(-\epsilon \sum_{1\le k\le m} f\left(\bar{x} + \epsilon\frac{k}{n}\right)\right) \tag{4.11}$$

$$\tilde{J}_4(n) := 2 + \sum_{\substack{1\le m\\ \epsilon=\pm 1}} \exp\left(-\sum_{1\le s} \frac{\epsilon^{s+1}}{n^s} \left(\mathfrak{b}_s^*(m) - \mathfrak{b}_s^*(0)\right) \bar{f}_s\right) \tag{4.12}$$

[32] It will merely change the *transasymptotics* by adding exponentially small summands. In terms of the Borel transforms $\hat{J}_{3,4}(\nu)$ or $\overset{\frown}{J}_{3,4}(\nu)$, it means that their nearest singularities will remain unchanged.

We can now regroup the factors $\tilde{J}_3$ and $\tilde{J}_4$ into $\tilde{J}_{3,4}$ and switch from the Bernoulli-type polynomials $\mathfrak{b}^*_s(m)$ over to their Euler-Bernoulli counterparts $\beta^*_s(m)$. These have the advantage of being odd/even function of m is s is even/odd, which will enable us to replace m-summation on $\mathbb{N}$ by m-summation on $\frac{1}{2}+\mathbb{Z}$, eventually easing the change from m-summation to τ-integration. Using the parity properties of $\beta^*(\tau)$ (see Section 3.1) we successively find:

$$\tilde{J}_{3,4}(n)=2\exp\left(-\sum_{1\leq s}\frac{1}{n^s}\mathfrak{b}_s\,\bar{f}_s\right)+\sum_{\substack{1\leq m\\ \epsilon=\pm1}}\exp\left(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\mathfrak{b}^*_s(m)\,\bar{f}_s\right) \tag{4.13}$$

$$\tilde{J}_{3,4}(n)\;=\sum_{\substack{0\leq m\\ \epsilon=\pm1}}\exp\left(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\mathfrak{b}^*_s(m)\,\bar{f}_s\right) \tag{4.14}$$

$$\tilde{J}_{3,4}(n)=\sum_{\substack{0\leq m\\ \epsilon=\pm1}}\exp\left(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\beta^*_s(m+\frac{1}{2})\,\bar{f}_s\right) \tag{4.15}$$

$$\tilde{J}_{3,4}(n)=\sum_{\substack{0\leq m\\ \epsilon=\pm1}}\exp\left(-\sum_{1\leq s}\frac{1}{n^s}\beta^*_s\left(\epsilon m+\epsilon\frac{1}{2}\right)\,\bar{f}_s\right)\qquad\text{(by parity!)} \tag{4.16}$$

$$\tilde{J}_{3,4}(n)=\sum_{m\,\in\,\frac{1}{2}+\mathbb{Z}}\exp\left(-\sum_{1\leq s}\frac{1}{n^s}\beta^*_s(m)\,\bar{f}_s\right). \tag{4.17}$$

This last identity should actually be construed as:

$$\tilde{J}_{3,4}(n)\;=\sum_{m\,\in\,\frac{1}{2}+\mathbb{Z}}\exp\left(-\frac{1}{n}\beta^*_1(m)\,\bar{f}_1\right)\exp_{\#}\left(-\sum_{2\leq s}\frac{1}{n^s}\beta^*_s(m)\,\bar{f}_s\right). \tag{4.18}$$

Here, the first exponential *exp* decreases fast as m grows, since

$$\beta^*_1(m)\,\bar{f}_1=\frac{1}{2}m^2\,\bar{f}_1=\frac{1}{2}m^2 f'(\bar{x})>0.$$

The second exponential $\exp_{\#}$, on the other hand, should be *expanded* as a power series of its argument and each of the resulting terms $m^{s_1}\,n^{-s_2}$ should be dealt with separately, leading to a string of clearly *convergent* series. We can now replace the *discrete* m-summation in (4.17) by a *continuous* τ-integration: here again, that may change the *transasymptotics* in n, but not the *asymptotics*.[33] We find, using the parity properties of the

[33] Indeed, the summation/integration bounds are $\pm\infty$, with a summand//integrand that vanishes exponentially fast there.

β_s^* and maintaining throughout the distinction between *exp* (unexpanded) and $exp_\#$ (expanded):

$$\tilde{J}_{3,4}(n) = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\right) \exp_\#\left(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\right) d\tau \tag{4.19}$$

$$\tilde{J}_{3,4}(n) = \sum_{\epsilon=\pm 1}\int_0^{+\infty} \exp\left(-\frac{1}{n}\beta_1^*(\epsilon\tau)\,\bar{f}_1\right) \exp_\#\left(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\epsilon\tau)\,\bar{f}_s\right) d\tau \tag{4.20}$$

$$\tilde{J}_{3,4}(n) = \sum_{\epsilon=\pm 1}\int_0^{+\infty} \exp\left(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\right) \exp_\#\left(-\sum_{2\le s}\frac{\epsilon^{s+1}}{n^s}\beta_s^*(\tau)\,\bar{f}_s\right) d\tau \tag{4.21}$$

$$\tilde{J}_{3,4}(n) = 2\int_0^{+\infty} \exp\left(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\right) \exp_\#\left(-\sum_{3\le s\ odd}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\right)$$

$$\times \cosh_\#\left(-\sum_{2\le s\text{ even}}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\right) d\tau \tag{4.22}$$

$$\tilde{J}_{3,4}(n) = 2\left[\int_0^{+\infty} \exp\left(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\right) \exp_\#\left(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\right) d\tau\right]_{\substack{\text{demi}\\\text{integ.}\\\text{part}}} \tag{4.23}$$

The notation in (4.23) means that we retain only the demi-integral powers of n^{-1} in $[\ldots]$. In view of the results of Section 2.3 about the correspondence between singularities and Taylor coefficient asymptotics, the Borel transform $\overset{\wedge}{\ell}\, i(\nu) := \hat{J}_{3,4}\,(\nu)$ of $\tilde{J}_{3,4}(n)$, or rather its counterpart $\hat{L}\, i(\zeta) := \overset{\wedge}{\ell}\, i(\log(1+\frac{\zeta}{\omega_F}))$ in the ζ-plane, must in our test-case (4.1) describe the closest singularities of the *sum-product* function $j(\zeta)$ or rather its "cleansed" variant $j^\#(\zeta)$.

Singularities such as Li shall be referred to as *inner generators* of the resurgence algebra. They differ from the three other types of generators (*original, exceptional, outer*) first and foremost by their stability: unlike these, they self-reproduce indefinitely under alien differentiation. Another difference is this: inner generators (minors and majors alike) tend to carry only *demi-integral*[34] powers of ζ or ν, as we just saw, whereas

[34] In our test-case, *i.e.* for a driving function f with a simple zero at $\bar{x}$. For zeros of odd order $\tau > 1$ (τ has to be odd to produce an extremum in f^*) we would get ramifications of order $(-\tau + 2s)/(\tau+1)$ with $s \in \mathbb{N}$, which again rules out entire powers. See Section 4.2-5 and also Section 6.1.

the other types of generators tend to carry only integral powers (in the minors) and logarithmic terms (in the majors).

So far, so good. But what about the two omitted factors $J_1(n)$ and $J_2(n)$? The second one, $J_2(n)$, which is a mere exponential $exp(-n\bar{\nu})$, simply accounts for the location $\bar{\zeta} = e^{\bar{\nu}}$ at which Li is seen in the ζ-plane. As for the ingress factor $J_1(n)$, *keeping it* (*i.e.* merging it with $J_{3,4}(n)$) would have rendered Li dependent on the ingress point $x = 0$, whereas *removing it* ensures that Li (and by extension the whole *inner algebra*) is totally independent of the 'accidents' of its construction, such as the choice of ingress point in the x-plane.

As for the move from $\{\mathfrak{b}_s^*\}$ to $\{\beta_s^*\}$, apart from easing the change from summation to integration, it brings another, even greater benefit: by removing the crucial coefficient β_0 in (3.10) (which vanishes, unlike $\mathfrak{b}_0$ in (3.3)), it shall enable us to express the future *mir*-transform as a purely integro-differential operator.[35]

One last remark, before bringing these heuristics to a close. We have chosen here the simplest possible way of producing an *inner generator*, namely directly from the *original generator i.e.* the sum-product series itself. To do this, rather stringent assumptions on the driving function f had to be made.[36] However, even when these assumptions are not met, the *original generator* always produces so-called *outer generators* (at least one, but generally two), which in turn always produce *inner generators*. So these two types – outer and inner – are a universal feature of sum-product series.

4.2 The long chain behind *nir*//*mir*

Let us now introduce two non-linear functional transforms central to this investigation. The *nir*-transform is directly inspired by the above heuristics. It splits into a chain of subtransforms, all of which are elementary, save for one: the *mir*-transform.

Both *nir* and *mir* depend on a coherent choice of scalars β_k and polynomials $\beta_k^*(\tau)$. The *standard choice*, or Euler-Bernoulli choice, corresponds to the definitions (3.9)-(3.15). It is the one that is relevant in most applications to analysis and SP-series. However, to gain a better insight into the β-dependence of *nir*//*mir*, it is also useful to consider the *non-standard choice*, with free coefficients β_k and accordingly redefined

[35] Indeed, if it β_0 didn't vanish, that lone coefficient would suffice to ruin nearly all the basic formulae (*infra*) about *mir* and *nir*.

[36] Like (4.1).

polynomials $\beta_k^*(\tau)$:

standard choice	non-standard choice
$\beta(\tau) := \dfrac{1}{e^{\tau/2} - e^{-\tau/2}} = \sum\limits_{-1 \le k} \beta_k\, \tau^k$	$\beta(\tau) := \sum\limits_{-1 \le k} \beta_k\, \tau^k$

$$\beta_k^*(\tau) := \beta(\partial_\tau)\, \tau^k = \frac{B_{k+1}\left(\tau + \frac{1}{2}\right)}{k+1}$$

$$\beta_k^*(\tau) := \beta(\partial_\tau)\, \tau^k = \sum_{s=-1}^{s=k} \beta_s\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)}.$$

As we shall see, even in the non-standard case it is often necessary to assume that $\beta_{-1} = 1$ and $\beta_0 = 0$ (like in the standard case) to get interesting results. The further coefficients, however, can be anything.

The long, nine-link chain:

$$\begin{array}{ccccccc}
 & \overset{\mathbf{nir}}{\rightarrow} & \rightarrow & \rightarrow & \rightarrow & \overset{\mathbf{nir}}{\rightarrow} & H \\
\uparrow & & & & & \downarrow & \uparrow^{9} \\
\uparrow & & f^* & \overset{3}{\rightarrow} & g^* & \downarrow & h' \\
\uparrow & & \uparrow^{2} & & \downarrow^{4} & \downarrow & \uparrow^{8} \\
 & \overset{\mathbf{nir}}{\longleftarrow} & f & & g & \overset{\mathbf{nir}}{\longrightarrow} & h \\
 & & \uparrow^{1} & & \downarrow^{5} & & \uparrow^{7} \\
 & & F & & \cancel{g} & \xrightarrow{\quad 6 \quad} & \hbar \\
 & & & & & \mathbf{mir}. &
\end{array}$$

$$H = \widehat{Li} \ \text{ or } \ \widehat{Le}$$

$$h' = \widehat{\ell i} \ \text{ or } \ \widehat{\ell e}$$

$$h = \overset{\frown}{\ell i} \ \text{ or } \ \overset{\frown}{\ell e}$$

Details of the nine steps:

$\overset{1}{\rightarrow}$:	precomposition :	$F \rightarrow f$	with	$f(x) := -\log F(x)$
$\overset{2}{\rightarrow}$:	integration :	$f \rightarrow f^*$	with	$f^*(x) := \int_0^x f(x_0)\, dx_0$
$\overset{3}{\rightarrow}$:	reciprocation :	$f^* \rightarrow g^*$	with	$f^* \circ g^* = id$
$\overset{4}{\rightarrow}$:	derivation :	$g^* \rightarrow g$	with	$g(y) := \dfrac{d}{dy} g^*(y)$
$\overset{5}{\rightarrow}$:	inversion :	$g \rightarrow \cancel{g}$	with	$\cancel{g}(y) := 1/g(y)$
$\overset{6}{\rightarrow}$:	**mir** functional :	$\cancel{g} \rightarrow \hbar$	with	int.-diff. expression below
$\overset{7}{\rightarrow}$:	inversion :	$\hbar \rightarrow h$	with	$h(\nu) := 1/\hbar(\nu)$
$\overset{8}{\rightarrow}$:	derivation :	$h \rightarrow h'$	with	$h'(\nu) := \dfrac{d}{d\nu} h(\nu)$
$\overset{9}{\rightarrow}$:	postcomposition :	$h' \rightarrow H$	with	$H(\zeta) := h'\left(\log\left(1 + \dfrac{\zeta}{\omega}\right)\right)$
$\overset{2...7}{\rightarrow}$:	**nir** functional :	$g \rightarrow h$	with	see Section 4.3 *infra*.

"Compact" and "layered" expansions of *mir*
The "sensitive" part of the nine-link chain, namely the *mir*-transform, is a non-linear integro-differential functional of infinite order. Pending its detailed description in Section 4-5, let us write down the general shape of its two expansions: the "compact" expansion, which merely isolates the r-linear parts, and the more precise "layered" expansion, which takes the differential order into account. We have:

$$\frac{1}{\hbar} := \frac{1}{\not g} + \sum_{1\leq r\,\in \text{odd}} \mathbb{H}_r(\not g) = \frac{1}{\not g} + \sum_{1\leq r\,\in \text{odd}} \partial^{1-r}\, \mathbb{D}_r(\not g) \quad (\text{“compact''})$$

$$\frac{1}{\hbar} := \frac{1}{\not g} + \sum_{\substack{1\leq r\,\in \text{odd}\\ \frac{1}{2}(r+1)\leq s\leq r}} \mathbb{H}_{r,s}(\not g) = \frac{1}{\not g} + \sum_{\substack{1\leq r\,\in \text{odd}\\ \frac{1}{2}(r+1)\leq s\leq r}} \partial^{-s}\, \mathbb{D}_{r,s}(\not g) \quad (\text{“layered''})$$

with r-linear, purely differential operators $\mathbb{D}_r$, $\mathbb{D}_{r,s}$ of the form

$$\mathbb{D}_r(\not g) := \sum_{\substack{\sum n_i = r\\ \sum i\, n_i = r-1}} {}^*\mathrm{Mir}^{n_0,n_1,\ldots,n_{r-1}} \prod_{0\leq i\leq r-1} (\not g^{(i)})^{n_i} \qquad (\text{“compact''})$$

$$\mathbb{D}_{r,s}(\not g) := \sum_{\substack{\sum n_i = r\\ \sum i\, n_i = s}} \mathrm{Mir}^{n_0,n_1,\ldots,n_s} \prod_{0\leq i\leq s} (\not g^{(i)})^{n_i} \qquad (\text{“layered''})$$

and connected by:

$$\partial\, \mathbb{D}_r(\not g) = \sum_{\frac{1}{2}(r+1)\leq s\leq r} \partial^{r+s}\, \mathbb{D}_{r,s}(\not g) \qquad (\forall r \in \{1,3,5\ldots\}).$$

The β-dependence is of course hidden in the definition of the differential operators $\mathbb{D}_r$, $\mathbb{D}_{r,s}$: *cf.* Section 4.5 *infra*. All the information about the *mir* transform is thus carried by the two rational-valued, integer-indexed moulds ${}^*\mathrm{Mir}$ and Mir.

4.3 The *nir* transform

Integral expression of *nir*
Starting from f we define $f^{\Uparrow\beta^*}$ and $f^{\uparrow\beta^*}$ as follows:

$$f(x) = \sum_{k\geq\kappa} f_k\, x^k \qquad (\kappa\geq 1\ ,\ f_\kappa\neq 0)$$

$$f^{\Uparrow\beta^*}(n,\tau) := \beta(\partial_\tau)\, f\left(\frac{\tau}{n}\right) \qquad \text{with} \quad \partial_\tau := \frac{\partial}{\partial\tau} \tag{4.24}$$

$$:= \sum_{k\geq\kappa} f_k\, n^{-k}\, \beta_k^*(\tau) \tag{4.25}$$

$$:= f_\kappa\, \frac{n^{-\kappa}\, \tau^{\kappa+1}}{\kappa+1} + f^{\uparrow\beta^*}(n,\tau). \tag{4.26}$$

These definitions apply in the standard and non-standard cases alike. Recall that in the standard case $\beta^*_k(\tau) = \frac{B_{k+1}(\tau+\frac{1}{2})}{k+1}$ is an even//odd function of τ for k odd//even, with leading term $\frac{\tau^{k+1}}{k+1}$.

The *nir*-transform $f \mapsto h$ is then defined as follows:

$$h(\nu)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n}\int_0^\infty \exp^{\#}(-f^{\Uparrow\beta^*}(n,\tau))\,d\tau \tag{4.27}$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n}\int_0^\infty \exp\Big(-f_\kappa\,\frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1}\Big)\exp_{\#}\big(-f^{\uparrow\beta^*}(n,\tau)\big)\,d\tau$$

where $\exp_{\#}(X)$ (respectively $\exp^{\#}(X)$) denotes the exponential expanded as a power series of X (respectively of X minus its leading term). Here, we first perform term-by-term, ramified Laplace integration in τ:

$$\int_0^\infty \exp\Big(-f_\kappa\,\frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1}\Big)\tau^p d\tau = (f_\kappa)^{-\frac{p+1}{\kappa+1}}(\kappa+1)^{\frac{p+1}{\kappa+1}-1}\Gamma\left(\frac{p+1}{\kappa+1}\right)n^{\frac{\kappa(p+1)}{\kappa+1}}$$

(with the main determination of $(f_\kappa)^{-\frac{p}{\kappa+1}}$ when $\Re(f_\kappa) > 0$) and then term-by-term (upper) Borel integration in n:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c-i\infty} e^{n\nu}n^{-q}\,\frac{dn}{n} = \frac{\nu^q}{q!}.$$

Lemma 4.1 (The *nir*-transform preserves convergence). *Starting from a* (*locally*) *convergent f, the $\boldsymbol{\tau}$-integration in* (4.27) *usually destroys convergence, but the subsequent* n*-integration always restores it. This holds not only in the standard case, but also in the non-standard one, provided $\beta(\tau)$ has positive convergence radius.*

4.4 The reciprocation transform

Let us first examine what becomes of the nine-link chain in the simplest non-standard case, *i.e.* with $\beta(\tau) := \tau^{-1}$.

Lemma 4.2 (The simplest instance of *nir*-transform). *For the choice $\beta(\tau) := \tau^{-1}$, the pair $\{h, \hbar\}$ coincides with the pair $\{g, \cancel{g}\}$. In other words,* mir *degenerates into the identity, and* nir *essentially reduces to changing the germ f^* into its functional inverse g^* (“reciprocation”).*

Since $g = h$, the third column in the “long chain” becomes redundant here, and the focus shifts to the first two columns, to which we adjoin a new entry $\cancel{f} := 1/f$ for the sake of symmetry. Lagrange's classical inversion formula fittingly describes the involutions $f^* \leftrightarrow g^*$ and $f \leftrightarrow$

g, and the simplest way of proving the above lemma is indeed by using Lagrange's formula. On its own, however, that formula gives no direct information about the involution $\cancel{f} \leftrightarrow \cancel{g}$ or the cross-correspondences $\cancel{f} \leftrightarrow g$ and $f \leftrightarrow \cancel{g}$ which are highly relevant to an understanding of the nine-link chain, including in the general case, *i.e.* for an arbitrary $\beta(\tau)$. So let us first redraw the nine-link chain in the "all-trivial case" $\{\beta(\tau) = \tau^{-1}, \kappa = 0, f(0) = g(0) = 1\}$ and then proceed with a description of the three afore-mentioned correspondences.

$$
\begin{array}{cccccccll}
\overset{\textbf{nir}}{\rightarrow} & \rightarrow & \rightarrow & \rightarrow & \overset{\textbf{nir}}{\rightarrow} & & H & (\textbf{Trivial case}: \ \boldsymbol{\beta(\tau)=\tau^{-1}}) & \\
\uparrow & & & \downarrow & \uparrow^{9} & & & & \\
\uparrow & f^{*} & \overset{3}{\rightarrow} & g^{*} & \downarrow & & g' & f^{*}(x)=x+a^{*}(x) & g^{*}(y)=y+b^{*}(y) \\
\uparrow & \uparrow^{2} & & \downarrow^{4} & \downarrow & & \uparrow^{8} & & \\
\overset{\textbf{nir}}{\longleftarrow} & f & & g & \overset{\textbf{nir}}{\longleftarrow} & \overset{\textbf{nir}}{\longrightarrow} & g & f(x)=1+a(x) & g(y)=1+b(y) \\
\nearrow^{1} & \uparrow & & \downarrow^{5} & & & \uparrow^{7} & & \\
F & \cancel{f} & & \cancel{g} & \longrightarrow & \overset{6}{\longrightarrow} & \cancel{g} & \cancel{f}(x)=1+\cancel{a}(x) & \cancel{g}(y)=1+\cancel{b}(y). \\
 & & & & \textbf{mir=id} & & & &
\end{array}
$$

Lemma 4.3 (Three variants of Lagrange's inversion formula). *The entries* a, b, $\cancel{\text{a}}$, $\cancel{\text{b}}$ *in the above diagram are connected by:*

$$a = \sum_{r\geq 1} b_{<r>} \quad \text{with } b_{<r>} = \sum_{n_1+\cdots+n_r=r} M^{n_1,\ldots,n_r}\, b^{[n_1,\ldots,n_r]} \tag{4.28}$$

$$a = \sum_{r\geq 1} \cancel{b}_{\{r\}} \quad \text{with } \cancel{b}_{\{r\}} = \sum_{n_1+\cdots+n_r=r} P^{n_1,\ldots,n_r}\, \cancel{b}^{[n_1,\ldots,n_r]} \tag{4.29}$$

$$\cancel{a} = \sum_{r\geq 1} \cancel{b}_{[r]} \quad \text{with } \cancel{b}_{[r]} = \sum_{n_1+\cdots+n_r=r} Q^{n_1,\ldots,n_r}\, \cancel{b}^{[n_1,\ldots,n_r]} \tag{4.30}$$

with differentially neutral[37] *and symmetral*[38] *integro-differential expressions* $\varphi^{[n_1,\ldots,n_r]}$ *defined as follows*:

$$\varphi^{[n_1,\ldots,n_r]}(t) := \int_{0<t_1<..<t_r<t} \varphi^{(n_1)}(t_1)\ldots\varphi^{(n_r)}(t_r)\, dt_1\ldots dt_r \tag{4.31}$$

---

[37] Indeed, since $n_1+\cdots+n_r=r$, we integrate as many times as we differentiate.

[38] Meaning that for any two sequences $\mathbf{n}'=(n_i')$ and $\mathbf{n}''=(n_i'')$, we have the multiplication rule $\varphi^{[\mathbf{n}']}\varphi^{[\mathbf{n}'']} \equiv \sum \varphi^{[\mathbf{n}]}$ with a sum running through all $\mathbf{n} \in$ *shuffle*$(\mathbf{n}', \mathbf{n}'')$.

and with scalar moulds $M^\bullet$, $P^\bullet$, $Q^\bullet$ easily inferred from the relations:

$$\sum_{\|\bullet\|=r} M^\bullet\, b^{[\bullet]} = \frac{(-1)^r}{r!}\partial^r\,(I\,b)^r \tag{4.32}$$

$$\sum_{\|\bullet\|=r} P^\bullet\, \cancel{b}^{[\bullet]} = \partial_R \cancel{b}\, I_L \ldots \partial_R \cancel{b}\, I_L \qquad (r \;\; \text{times}) \tag{4.33}$$

$$1 + \sum_\bullet Q^\bullet \cancel{b}^{[\bullet]} = \left(1 + \sum_\bullet P^\bullet \cancel{b}^{[\bullet]}\right)^{-1}. \tag{4.34}$$

Remark 4.1. In (4.32), ∂ as usual stands for differentiation and $I = \partial^{-1}$ for integration from 0. In (4.33), ∂_R denotes the differentiation operator *acting on everything to its right* and $I_L = \partial_L^{-1}$ denotes the integration operator (with integration starting, again, from 0) *acting on everything to its left*.

Thus we find:

$$\begin{aligned}
a &= b_{<1>} + b_{<2>} + b_{<3>} + \ldots \\
b_{<1>} &= -b^{[1]} \\
b_{<2>} &= +b^{[0,2]} + b^{[1,1]} \\
b_{<3>} &= -b^{[0,0,3]} - 4\,b^{[0,1,2]} - 4\,b^{[1,0,2]} - 3\,b^{[0,2,1]} - 15\,b^{[1,1,1]} \\
&\ldots
\end{aligned}$$

$$\begin{aligned}
a &= \cancel{b}_{\{1\}} + \cancel{b}_{\{2\}} + \cancel{b}_{\{3\}} + \ldots \\
\cancel{b}_{\{1\}} &= +\cancel{b}^{[1]} \\
\cancel{b}_{\{2\}} &= +\cancel{b}^{[0,2]} + \cancel{b}^{[1,1]} \\
\cancel{b}_{\{3\}} &= +\cancel{b}^{[0,0,3]} + 2\,\cancel{b}^{[0,1,2]} + \cancel{b}^{[1,0,2]} + \cancel{b}^{[0,2,1]} + \cancel{b}^{[1,1,1]} \\
&\ldots
\end{aligned}$$

$$\begin{aligned}
\cancel{a} &= \cancel{b}_{[1]} + \cancel{b}_{[2]} + \cancel{b}_{[3]} + \ldots \\
\cancel{b}_{[1]} &= -\cancel{b}^{[1]} \\
\cancel{b}_{[2]} &= +\cancel{b}^{[1,1]} - \cancel{b}^{[0,2]} \\
\cancel{b}_{[3]} &= -\cancel{b}^{[1,1,1]} - \cancel{b}^{[0,0,3]} + \cancel{b}^{[0,2,1]} + \cancel{b}^{[1,0,2]} \\
&\ldots
\end{aligned}$$

Remark 4.2. The coefficients $M^\bullet$, $P^\bullet$, $Q^\bullet$ verify the following identities, all of which are elementary, save for the last one (involving $\sum |Q^\bullet|$):

$$\sum_{\|\bullet\|=r} (-1)^r\, M^\bullet = \sum_{\|\bullet\|=r} |M^\bullet| = r^r \tag{4.35}$$

$$\sum_{\|\bullet\|=r} P^\bullet = \sum_{\|\bullet\|=r} |P^\bullet| = r! \tag{4.36}$$

$$\sum_{\|\bullet\|=r\geq 2} Q^\bullet = 0, \qquad \sum_{\|\bullet\|=r\geq 2} |Q^\bullet| = \frac{(2r-1)!}{(r-1)!\,r!}. \tag{4.37}$$

Remark 4.3. a in terms of b is an elementary consequence of Lagrange's formula for functional inversion, but a in terms of $\cancel{b}$ and $\cancel{a}$ in terms of $\cancel{b}$ are not.

Remark 4.4. The formulas (4.28) through (4.30) involve only *sublinear* sequences $\mathbf{n} = \{n_1, \ldots, n_r\,;\ n_i \geq 0\}$, *i.e.* sequences verifying:

$$n_1 + \cdots + n_i \leq i \qquad \forall i \qquad \text{and} \quad n_1 + \cdots + n_r = r. \tag{4.38}$$

The number of such series is exactly $\frac{(2r)!}{r!(r+1)!}$ (Catalan number), which puts them in one-to-one correspondence with r-node binary trees. Moreover, these sublinear sequences are stable under *shuffling* and this establishes a link with the "classical product" on binary trees.[39]

Remark 4.5. The various $\varphi^{[n_1,\ldots,n_r]}$, even for sublinear sequences $[n_1, \ldots, n_r]$, are not linearly independent, but this does not detract from the canonicity of the expansions in (4.28), (4.29), (4.30) because the induction rules (4.32), (4.33), (4.34) behind the definition of $M^\bullet$, $P^\bullet$, $Q^\bullet$ unambiguously define a privileged set of coefficients.

4.5 The *mir* transform

Lemma 4.4 (Formula for *mir* in the standard case). *The mir transforms $\cancel{g} \mapsto \hbar$ is explicitly given by:*

$$\frac{1}{\hbar(\nu)} = \left[\frac{1}{\cancel{g}(\nu)} \exp_\# \left(-\sum_{r\geq 1} \beta_r\, I^r\, (\cancel{g}(\nu)\partial_\nu)^r\, \cancel{g}(\nu) \right) \right]_{I=\partial_\nu^{-1}}. \tag{4.39}$$

[39] In fact, that product is of recent introduction: see Loday, Ronco, Novelli, Thibon, Hivert in [13] and [14].

Mind the proper sequence of operations:

– *first*, we expand the blocks $(\not{g}(\nu)\partial_\nu)^r \not{g}(\nu)$.

– *second*, we expand $\exp_{\#}(\dots)$, which involves taking the suitable powers of the formal variable I (with "I" standing for *integration*).

– *third* , we divide by $\not{g}(\nu)$.

– *fourth*, we move each I^r to the left-most position.[40]

– *fifth*, we replace each I^r by the operator ∂_ν^{-r} which stands for n successive formal integrations from 0 to ν.

– *sixth*, we carry out these integrations.

Lemma 4.5 (The integro-differential components $\mathbb{D}_{r,s}$ of *mir*). *The* mir *functional admits a canonical expansion*:

$$\frac{1}{\hbar} := \frac{1}{\not{g}} + \sum_{\substack{1 \le r\, \in \text{odd} \\ \frac{1}{2}(r-1)\le s\le r}} \mathbb{H}_{r,s}(\not{g}) = \frac{1}{\not{g}} + \sum_{\substack{1 \le r\, \in \text{odd} \\ \frac{1}{2}(r-1)\le s\le r}} \partial^{-s}\, \mathbb{D}_{r,s}(\not{g}) \tag{4.40}$$

with r-linear differential operators $\mathbb{D}_{r,s}$ of total order d:

$$\mathbb{D}_{r,s}(\not{g}) := \sum_{\substack{\sum n_i = r \\ \sum i\, n_i = s}} \mathrm{Mir}^{n_0,n_1,\dots,n_s} \prod_{0\le i\le s} (\not{g}^{(i)})^{n_i} \tag{4.41}$$

and coefficients $\mathrm{Mir}^{n_0,n_1,\dots,n_s} \in \frac{1}{s!}\,\mathbb{Z}[\beta_1,\beta_2,\beta_3,\dots]$ *which are themselves homogeneous of "degree"* $\mathrm{r}+1$ *and "order"* s *if to each β_i we assign the "degree"* $\mathrm{i}+1$ *and "order"* i.

For the standard choice $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$, we have $0 = \beta_2 = \beta_4 = \dots$, and so we get only integro-differential components $\mathbb{D}_{r,s}$ which have

[40] This means that all the powers of $\not{g}$, $\not{g}'$, $\not{g}''$ etc. must be put to the right of I^r.

all *odd* degrees $r = 1, 3, 5 \ldots$. Thus:

$$
\begin{aligned}
\mathbb{D}_{1,1} &= +\frac{1}{24}\big(\not{g}'\big)\\
\mathbb{D}_{3,2} &= +\frac{1}{1152}\big(\not{g}\,\not{g}'^{2}\big)\\
\mathbb{D}_{3,3} &= -\frac{7}{5760}\big(\not{g}'^{3} + \not{g}^{2}\not{g}''' + 4\,\not{g}\,\not{g}'\not{g}''\big)\\
\mathbb{D}_{5,3} &= +\frac{1}{82944}\big(\not{g}^{2}\not{g}'^{3}\big)\\
\mathbb{D}_{5,4} &= -\frac{7}{138240}\big(\not{g}\,\not{g}'^{4} + \not{g}^{3}\not{g}'\not{g}''' + 4\,\not{g}^{2}\not{g}'^{3}\big)\\
\mathbb{D}_{5,5} &= +\frac{31}{967680}\big(\not{g}'^{5} + \not{g}^{4}\not{g}^{(5)} + 11\,\not{g}^{3}\not{g}'\not{g}^{(4)} + 32\,\not{g}^{2}\not{g}'^{2}\not{g}'''\\
&\qquad + 15\,\not{g}^{3}\not{g}''\not{g}''' + 26\,\not{g}\,\not{g}'^{3}\not{g}'' + 34\,\not{g}^{2}\not{g}'\not{g}''^{2}\big).
\end{aligned}
$$

However the *mir* formula has a wider range:

Lemma 4.6 (Formula for *mir* in the non-standard case). *The formula* (4.39) *and* (4.40) *for* mir *remains valid if we replace* $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ *by any series of the form* $\beta(\tau) := \sum_{n\geq -1} \beta_n\, \tau^n$ *subject to* $\beta_{-1} = 1$, $\beta_0 = 0$. *The even-indexed coefficients* β_{2n} *need not vanish. When they don't, the expansion* (4.40) *may involves homogeneous components* $\mathbb{H}_{r,s}$ *of any degree* r, *odd or even.*

Dropping the condition $\beta_{-1} = 1$ would bring about only minimal changes, but allowing a non-vanishing β_0 would deeply alter and complicate the shape of the *mir* transform: it would cease to be a purely integro-differential functional. We must therefore be thankful for the parity phenomenon (see Section 4.1 *supra*) responsible for the occurence, in the *nir* integral, of the Bernoulli polynomials with shift $1/2$ rather than 1.

Lemma 4.7 (Alternative interpretation for the *mir* formula). *The procedure implicit in formula* (4.39) *can be rephrased as follows*:

(i) *Form* $h(w, y) := \sum_{r\geq 1} \frac{w^r}{r!}\, (\not{g}(y)\partial_y)^r.y$;
(ii) *Form* $k(w, y) := \sum_{r\geq 1} \beta_r \frac{w^r}{r!}\, (\not{g}(y)\partial_y)^r.\not{g}(y)$;
(iii) *Interpret* $\not{g}(y)\partial_y$ *as an infinitesimal generator and* $h^{\circ w}(y) = h(w, y) = g^*(w + f^*(y))$ *as the corresponding group of iterates:* $h^{\circ w_1} \circ h^{\circ w_2} = h^{\circ(w_1+w_2)}$;
(iv) *Interpret* $k(w, y)$ *as the Hadamard product, with respect to the* w *variable, of* $\beta(w)$ *and* $\partial_w h(w, y)$;

(v) *Calculate the convolution exponential* $K(w,y) := \exp_\star(-k(w,y))$ *relative to the unit-preserving convolution* $\star$ *acting on the* w *variable*;
(vi) *Integrate* $\int_0^\nu K(\nu-\nu_1,\nu_1)(\cancel{g}(\nu_1))^{-1}\,d\nu_1 =: \ell(\nu)$.

4.6 Translocation of the *nir* transform

If we set $\eta := \int_0^\epsilon f(x)\,dx$ and then wish to compare:

(i) $f(x)$ and its translates ${}^\epsilon f(x) = e^{\epsilon\partial_x} f(x) = f(x+\epsilon)$;
(ii) $h(\nu)$ and its translates ${}^\eta h(\nu) = e^{\eta\partial_\nu} h(\nu) = h(\nu+\eta)$;

there are *a priori* four possibilities to choose from:

$$\begin{aligned}
&\text{choice 1}: \ (e^{\eta\partial_\nu}\,\mathrm{nir} - \mathrm{nir}\,e^{\epsilon\partial_x})\,f \ \text{ as a function of } \ (\epsilon, f)\\
&\text{choice 2}: \ (e^{\eta\partial_\nu}\,\mathrm{nir} - \mathrm{nir}\,e^{\epsilon\partial_x})\,f \ \text{ as a function of } \ (\eta, f)\\
&\text{choice 3}: \ (\mathrm{nir} - e^{-\eta\partial_\nu}\,\mathrm{nir}\,e^{\epsilon\partial_x})\,f \ \text{ as a function of } \ (\epsilon, f)\\
&\text{choice 4}: \ (\mathrm{nir} - e^{-\eta\partial_\nu}\,\mathrm{nir}\,e^{\epsilon\partial_x})\,f \ \text{ as a function of } \ (\eta, f).
\end{aligned}$$

In the event, however, the best option turns out to be choice 3. So let us define the finite (respectively infinitesimal) increments ∇h (respectively $\delta_m h$) accordingly:

$$\nabla h(\epsilon,\nu) = \sum (\delta_m h)(\nu)\,\epsilon^m := \mathrm{nir}(f)(\nu) - \mathrm{nir}({}^\epsilon f)(\nu-\eta) \tag{4.42}$$

$$\text{with} \qquad {}^\epsilon f(x) := f(x+\epsilon) \quad \text{and} \quad \eta := \int_0^\epsilon f(x)\,dx. \tag{4.43}$$

Going back to Section 4.3, we can calculate $\mathrm{nir}(f)(\nu)$ by means of the familiar double integral (4.27), and then $\mathrm{nir}({}^\epsilon f)(\nu-\eta)$ by using that same double integral, but after carrying out the substitutions:

$$\nu \mapsto \nu-\eta = \nu - \sum_{k\ge 0} f^{(k)}(0)\,\frac{\epsilon^{k+1}}{(k+1)!} = \nu - \sum_{k\ge 0} f_k\,\frac{\epsilon^{k+1}}{k+1} \tag{4.44}$$

$$f(x) \mapsto {}^\epsilon f(x) = \sum_{k_1,k_2\ge 0} f_{k_1+k_2}\,\frac{(k_1+k_2)!}{k_1!k_2!}\,\epsilon^{k_1}\,x^{k_2} \tag{4.45}$$

$$f^{\Uparrow\beta}(n,\tau) \mapsto {}^\epsilon f^{\Uparrow\beta}(n,\tau) = \sum_{k_1,k_2\ge 0} f_{k_1+k_2}\,\frac{(k_1+k_2)!}{k_1!k_2!}\,\epsilon^{k_1}\,n^{-k_2}\,\beta^*_{k_2}(\tau). \tag{4.46}$$

Singling out the contribution of the various powers of ϵ, we see that each infinitesimal increment $\delta_m h(\nu)$ is, once again, given by the double *nir*-integral, the only difference being that the integrand must now be multiplied by an elementary factor $D_m(n,\tau)$ polynomial in n^{-1} and τ. Massive cancellations occur, which wouldn't occur under any of the other choices 1, 2 or 4, and we can then regroup all the *infinitesimal* increments $\delta_m h(\nu)$ into one global and remarkably simple expressions for the *finite* increment:

Lemma 4.8 (The finite increment ∇h: compact expression). *Like* nir *itself, its finite increment is given by a double integral:*

$$\nabla h(\epsilon,\nu)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\int_0^{\epsilon\, n} \exp_{\#}\Big(-\beta(\partial_\tau)\, f\left(\frac{\tau}{n}\right)\Big)\, d\tau \tag{4.47}$$

but with truncated Laplace integral and with $\exp_{\#}$ *instead of* $\exp^{\#}$.

The presence in (4.47) of $\exp_{\#}$ instead of $\exp^{\#}$ means that we must now expand *everything* within exp, including the leading term (unlike in (4.27)). So we no longer have proper Laplace integration here. Still, due to the truncation $\int_0^{\epsilon n}(\dots)d\tau$ of the integration interval, the integral continues to make sense, at least term-by-term. Due to the form ϵn of the upper bound, it yields infinitely many summands n^{-s}, with positive *and* negative s. However, the second integration $\int_{c-i\infty}^{c+i\infty}(\dots)\frac{dn}{n}$ kill off the n^{-s} with negative s, and turns those with positive s into $\frac{\nu^s}{s!}$. If we correctly interpret and carefully execute the above procedure, we are led to the following analytical expressions for the increment:

Lemma 4.9 (The finite increment ∇h: analytical expression). *We have*:

$$\nabla h(\epsilon,\nu):=\sum_{s\geq 1}\frac{(-1)^s}{s!}\sum_{\substack{p_i\geq 0\,,\ p_i\geq q_i\\ q_i\geq -1\,,\ q_i\neq 0}}\frac{\epsilon^m}{m}\ \frac{\nu^n}{n!}\ \prod_{i=1}^{i=s}\left(f_{p_i}\beta_{q_i}\frac{p_i!}{(p_i-q_i)!}\right) \tag{4.48}$$

$$\textit{with}\ \ m:=1+\sum_i p_i-\sum_i q_i,\quad n:=-1+\sum_i q_i. \tag{4.49}$$

Equivalently, we may write:

$$\nabla h(\epsilon,\nu):=\sum_{\substack{m\geq 1\\ n\geq 1}}\delta_{m,n}(f,\beta)\ \epsilon^m\ \nu^n\qquad \textit{with} \tag{4.50}$$

$$\delta_{m,n}(f,\beta)=\sum_{s=1}^{m+n}\frac{(-1)^s}{m\,n!\,s!}\sum_{\substack{m_i\geq 0,\ m_i\geq -n_i\\ n_i\neq 0,\ n_i\geq -1}}\ \sum_{\substack{\sum m_i=m-1\\ \sum n_i=n+1}}\ \prod_{i=1}^{i=s}\left(f_{m_i+n_i}\,\beta_{n_i}\frac{(m_i+n_i)!}{m_i!}\right).$$

Let us now examine the *infinitesimal* increments $\delta_m h$ of (4.42). Their *analytical* expression clearly follows from (4.50), but they also admit very useful *compact* expressions. To write these down, we require two sets of power series, the $f^{\sharp m}$ and their upper Borel transforms $\overset{\frown}{f}^{\sharp m}$. These series enter the τ-expansion of $f^{\Uparrow\beta}$:

$$f^{\Uparrow\beta}(n,\tau) = f^{\sharp 0}(n) + \tau\, f^{\sharp 1}(n) + \tau^2\, f^{\sharp 2}(n) + \ldots \tag{4.51}$$

As a consequence, they depend bilinearly on the coefficients of f and β:

$$f^{\sharp 0}(n) := \sum_{0\le p} p!\, f_p \beta_p\, n^{-p} \qquad \overset{\frown}{f}^{\sharp 0}(\nu) := \sum_{0\le p} f_p\beta_p\, \nu^p \qquad (r=0)$$

$$f^{\sharp m}(n) := \sum_{m-1\le p} \frac{p!}{m!} f_p \beta_{p-m}\, n^{-p} \qquad \overset{\frown}{f}^{\sharp 0}(\nu) := \sum_{m-1\le p} \frac{p!}{m!} f_p\beta_{p-m}\, \nu^p \qquad (r\ge 1).$$

We also require the "upper" variant $\overline{*}$ of the finite-path convolution:

$$(A\overline{*}B)(t) := \int_0^t A(t-t_1)dB(t_1) = \int_0^t B(t-t_1)\, dA(t_1) \tag{4.52}$$

$$1\overline{*}1 \equiv 1, \qquad \frac{(.)^p}{p!}\overline{*}\frac{(.)^q}{q!} \equiv \frac{(.)^{p+q}}{(p+q)!} \tag{4.53}$$

along with the corresponding convolution exponential $\exp_{\overline{\star}}$:

$$\exp_{\overline{*}} A := 1 + A + \frac{1}{2} A\overline{*}A + \frac{1}{6} A\overline{*}A\overline{*}A + \ldots \tag{4.54}$$

Lemma 4.10 (Infinitesimal increments $\delta_m h$: compact expression). *The infinitesimal increments $\delta_m h$, as defined by the ϵ-expansion $\nabla h(\epsilon,\nu) = \sum_{0\le m}\epsilon^m\,(\delta_m h)(\nu)$, admit the compact expressions*:

$$\delta_1 h = \partial_\nu \exp_{\overline{*}}(-f^{\sharp 0}) \tag{4.55}$$

$$\delta_2 h = \frac{1}{2}\,\partial_\nu^2\Big(\big(-f^{\sharp 1}\big)\star \exp_\star(-f^{\sharp 0})\Big) \tag{4.56}$$

$$\delta_3 h = \frac{1}{3}\,\partial_\nu^3\left(\left(-f^{\sharp 2}+\frac{1}{2}(-f^{\sharp 1})\overline{*}(-f^{\sharp 1})\right)\overline{*}\exp_{\overline{*}}(-f^{\sharp 0})\right) \tag{4.57}$$

$$\ldots \quad \ldots$$

$$\delta_m h = \frac{1}{m}\,\partial_\nu^m\left(\left(\sum_{\substack{\sum_i k_i = m-1\\ i\ge 1}} \prod \frac{(-f^{\sharp i})^{\overline{*}k_i}}{k_i!}\right)\overline{*}\exp_{\overline{*}}(-f^{\sharp 0})\right). \tag{4.58}$$

Lemma 4.11 (The increments in the non-standard case). *The above expressions for $\delta_m h$ and ∇h remain valid even if we replace $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ by any series of the form $\beta(\tau) := \sum_{n\geq -1} \beta_n \tau^n$ subject only to $\beta_{-1} = 1, \beta_0 = 0$.*

Lemma 4.12 (Entireness of $\delta_m h$ and ∇h). *For any polynomial or entire input f, each $\delta_m h(\nu)$ is an entire function of ν and $\nabla h(\epsilon, \nu)$ is an entire function of (ϵ, ν). This holds not only for the standard choice $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ but also for any series $\beta(\tau) := \sum_{n\geq -1} \beta_n \tau^n$ with positive convergence radius.*[41]

This extremely useful lemma actually results from a sharper statement:

Lemma 4.13 (∇h bounded in terms of f and β). *If $f(x) \prec \frac{A}{1-ax}$ and $\beta(\tau) \prec \frac{B}{1-b\tau}$ then $\nabla h(\epsilon, \nu) \prec \frac{Const}{1-2a\epsilon\nu} \exp\left(\frac{2AB}{b}\frac{\epsilon}{(1-ab\nu)}\right)$.*

Here, of course, for any two power series $\{\varphi, \psi\}$, the notation $\varphi \prec \psi$ is short-hand for "ψ dominates φ", *i.e.* $|\varphi_n| \leq \psi_n \ \forall n$. Under the assumption $f(x) \prec \frac{A}{1-ax}$ and $\beta(\tau) \prec \frac{B}{1-b\tau}$ we get:

$$\overset{\frown\#0}{f}(\nu) \prec K_0(\nu) := \frac{AB}{1-ab\,\nu}$$

$$\overset{\frown\#m}{f}(\nu) \prec K_m(\nu) := \frac{AB}{ab}\frac{a^m}{m!}\frac{\nu^{m-1}}{1-ab\,\nu}$$

After some easy majorisations, this leads to:

$$\begin{aligned}\delta_m h(\nu) &\prec \partial_\nu^m \sum_{\substack{1\leq s\leq m \\ m_1+\cdots+m_s=m}} \frac{1}{s!}\big(K_{m_1}\overline{*}K_{m_2}\overline{*}\ldots K_{m_s}\big)(\nu) \\ &\prec \sum_{1\leq s\leq m} \frac{\text{Const}}{s!}\left(\frac{AB}{ab}\right)^s \frac{(2a)^r\,\nu^{r-s}}{(1-ab\nu)^s}\end{aligned}$$

and eventually to:

$$\begin{aligned}\nabla h(\epsilon, \nu) &\prec \frac{\text{Const}}{s!}\left(\frac{AB}{ab}\right)^s \frac{1}{(1-ab\nu)^s}\sum_{s\leq m}(2a\,\epsilon)^m\,\nu^{m-s} \\ &\prec \frac{\text{Const}}{1-2a\epsilon\nu}\exp\left(\frac{2AB}{b}\frac{\epsilon}{(1-ab\nu)}\right).\end{aligned}$$

[41] Subject as usual to $\beta_{-1} = 1, \beta_0 = 0$.

In the standard case we may take $B = 1, b = \frac{1}{2\pi}$ so that the bound becomes:

$$\nabla h(\epsilon, \nu) \prec \frac{\text{Cons}}{1 - 2a\,\epsilon\,\nu} \exp\left(\frac{4\pi\,A\,\epsilon}{1 - \dfrac{a}{2\pi}\,\nu}\right).$$

Since *Const* is independent of a, A, this immediately implies that $\nabla h(\epsilon,\nu)$ is bi-entire (in ϵ and ν) if $f(x)$ is entire in x.[42]

4.7 Alternative factorisations of *nir*. The *lir* transform

The *nir* transform and its two factorisations

In some applications, two alternative factorisations of the *nir*-transform are preferable to the one corresponding to the nine-link chain of Section 4.2. Graphically:

$$f \longrightarrow \xrightarrow{\textbf{nir}} \longrightarrow h$$

$$\begin{array}{ccccc} f^* & \xrightarrow{\textbf{rec}} & g^* & \xrightarrow{\textbf{imir}} & h^* \\ \uparrow & & \downarrow & & \downarrow \\ f & & g & \dots & h \\ & & \downarrow & & \uparrow \\ & & \cancel{g} & \xrightarrow{\textbf{mir}} & \hbar \end{array} \qquad \begin{array}{c} (f^* := \partial^{-1} f, g^* := \partial^{-1} g) \\ \\ \\ \\ (\cancel{g} := 1/g \ \ , \ \ \hbar := 1/g) \end{array}$$

$$\begin{array}{ccccc} f^* & \xrightarrow{\textbf{ilir}} & q^* & \xrightarrow{\textbf{rec}} & h^* \\ \uparrow & & \uparrow & & \downarrow \\ f & \xrightarrow{\textbf{lir}} & q & & h \end{array} \qquad \begin{array}{c} (q^* := \partial^{-1} q\ , h^* := \partial^{-1} h). \\ \\ \\ \end{array}$$

In the first alternative, we go by *imir* ("integral" *mir*) from the indefinite integral g^* of g to the indefinite integral h^* of h, rather than from $\cancel{g}$ to $\hbar$. In the second alternative, the middle column (g, g^*) gets replaced by (q, q^*) with q^* denoting the functional inverse of h^*. In that last scenario, the non-elementary factor-transform becomes *lir* (from f to q) or *ilir* (from f^* to q^*). We get get for these transforms expansions similar

[42] And *only* if f is entire - but this part is harder to prove and not required in practice.

to, but in some respects simpler than, the expansions for *mir*:

$$
\begin{aligned}
\mathrm{mir}:\ \not{g} \to \hbar \quad &\text{with } \frac{1}{\hbar} = \frac{1}{\not{g}} + \sum_{1 \le r \text{ odd}} \mathbb{H}_r(\not{g}) \\
\mathrm{lir}:\ f \to q \quad &\text{with } q = f + \sum_{3 \le r \text{ odd}} \mathbb{Q}_r(f)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{imir}:\ g^* \to h^* \quad &\text{with } h^* = g^* + \sum_{1 \le r \text{ odd}} \mathbb{IH}_r(\not{g}) \\
\mathrm{ilir}:\ f^* \to q^* \quad &\text{with } q^* = f^* + \sum_{3 \le r \text{ odd}} \mathbb{IQ}_r(f).
\end{aligned}
$$

Each term on the right-hand sides is a polynomial in the $f^{(i)}$ and the following integro-differential expressions:

$$
\begin{aligned}
f_m^{(d)\{s_1,\ldots,s_r\}} &:= (I_R \bullet f)^{s-d} \bullet (f^{-1} f^{(s_1)} \ldots f^{(s_r)}) && (4.59)\\
&= (I_R \bullet f)^{m-r} \bullet I_R \bullet (f^{(s_1)} \ldots f^{(s_r)}) && (4.60)\\
&= I_R \bullet f \bullet I_R \bullet f \ldots I_R \bullet f \bullet I_R \bullet (f^{(s_1)} \ldots f^{(s_r)})
\end{aligned}
$$

with $d \ge -1, m \ge r, s_1, \ldots, s_r \ge 1$ and $1+m+d = r+s$. Here $I_R := \partial^{-1} = \int_0^{\cdots}$ denotes the integration operator that starts from 0 and acts on *everything standing on the right*. The "monomial" $f_m^{(d)\{s_1,\ldots,s_r\}}$ has total degree m (*i.e.* it is m-linear in f) and total differential order d. The notation is slightly redundant since $1+m+d \equiv r+s \equiv \sum(1+s_i)$ but very convenient, since it makes it easy to check that each summand in the expression of $\mathbb{H}_r(f)$ (respectively $\mathbb{IH}_r(f)$) has global degree r and global order 0 (respectively -1). The operators $\mathbb{IH}_r$ and $\mathbb{IQ}_r$ are simpler and in a sense more basic than the $\mathbb{H}_r$ and $\mathbb{Q}_r$.

Proof. Let us write the two reciprocal (formal) functions h^* (known) and q^* (unknown) as sums of a leading term plus a perturbation:

$$
\begin{aligned}
h^*(x) &= g^*(x) + \mathbb{IH}(x)\\
q^*(x) &= f^*(x) + \mathbb{IQ}(x).
\end{aligned}
$$

The identity $id = h^* \circ q^*$ may be expressed as:

$$
\begin{aligned}
id &= (g^* + \mathbb{IH}) \circ (f^* + \mathbb{IQ})\\
&= id + \mathbb{IH} \circ f^* + \sum_{1 \le r} \frac{1}{r!} (\mathbb{IQ})^r \left(\partial^r (g^* + \mathbb{IH})\right) \circ f^*.
\end{aligned}
$$

But h^* may be written as

$$h^*(x) = (x + \mathbb{JH}) \circ g^*(x) \tag{4.61}$$

and the identity $id = h^* \circ q^*$ now becomes:

$$0 = \mathbb{JH} + \sum_{1 \le r} \frac{1}{r!} \, (\mathbb{IQ})^r \, (f^{-1}\!\bullet \partial)^r \bullet (x + \mathbb{JH}). \tag{4.62}$$

The benefit from changing $\mathbb{IH}$ into $\mathbb{JH}$ is that we are now handling direct functions of f. Indeed, in view of the argument in Section 4.2 we have:

$$\mathbb{JH} = \sum_{1 \le r, 1 \le s_i} (-1)^r \left(\prod_{i=1}^{i=r} \beta_i \right) (I_R \bullet f)^{1+\sum s_i} \bullet \left(f^{-1} \prod_{i=1}^{i=r} f^{(s_i)} \right). \tag{4.63}$$

The right-hand side turns out to be a linear combination of monomials (4.59) of order $d = -1$:

$$\mathbb{JH} = \sum_{1 \le r, 1 \le s_i} (-1)^r \left(\prod_{i=1}^{i=r} \beta_i \right) f_{r+\sum s_i}^{(-1)\{s_1,\ldots,s_r\}}. \tag{4.64}$$

If we now adduce the obvious rules for differentiating these monomials:

$$\begin{aligned}
(f^{-1}\!\bullet \partial)^\delta \bullet f_m^{(d)\{s_1,\ldots,s_r\}} &= f_{m-\delta}^{(d+\delta)\{s_1,\ldots,s_r\}} && (\textit{if } \delta \le m-r) \\
&= f^{-1} \, f^{(s_1)} \ldots f^{(s_1)} && (\text{if } \delta = 1+m-r) \\
&= (f^{-1}\!\bullet \partial)^{\delta+r-m-1}\!\bullet f^{-1} \, f^{(s_1)} \ldots f^{(s_1)} \\
& && (\text{if } \delta \ge 2+m-r)
\end{aligned}$$

we see at once that the identity (4.62) yields an inductive rule for calculating, for each m, the m-linear part $\mathbb{IQ}_m$ of $\mathbb{IQ}$. At the same time, it shows that any such $\mathbb{IQ}_m$ will be exactly of global differential order -1, and *a priori* expressible as a polynomial in f^{-1}, f, $f^{(1)}$, $f^{(2)}$, $f^{(3)} \ldots$ and finitely many monomials $f_\mu^{(\delta)\{s_1,\ldots,s_\rho\}}$. The only point left to check is the *non-occurence* of negative powers of f, which would seem to result from the above differentiation rules, but actually cancel out in the end result.

4.8 Application: kernel of the *nir* transform

For any input f of the form $p \, log(x) + \mathrm{Reg}(x)$ with $p \in \mathbb{Z}$ and Reg a regular analytic germ, the image h of f under *nir* is also a regular analytic germ:

$$nir: \quad f(x) = p \, \log(x) + \mathrm{Reg}_1(x) \ \mapsto \mathrm{Reg}_2(x).$$

The singular part of h, which alone has intrinsic significance, is thus 0. In other words, germs f with logarithmic singularities that are *entire* multiples of $\log(x)$ belong to the kernel of *nir* and produce *no inner generators*. This important and totally non-trivial fact is essential when it comes to describing the inner algebra of SP series j_F constructed from a meromorphic F. It may be proven (see [15]) either by using the alternative factorisations of the *nir* transform mentioned in the preceding subsection, or by using an exceptional generator $f(x-x_0)$ with base-point x_0 arbitrarily close to 0. An alternative proof, valid in the special case when $\mathrm{Reg}_1 = 0$ and relying on the existence in that case of a simple ODE for the *nir*-transform, shall be given in Section 6.6-7 below.

4.9 Comparing/extending/inverting *nir* and *mir*

Lemma 4.14 (The case of generalised power-series f). *The* nir *transform can be extended to generalised power series*

$$f(x) := \sum_{k_i \geq m} f_{k_i}\, x^{k_i} \quad \left(k_i \uparrow +\infty;\ k_i \in \mathbb{R} \dot{-} \{-1\}\right) \tag{4.65}$$

in a consistent manner (*i.e. one that agrees with* mir *and ensures that* $\ell(\nu)$ *converges whenever* $f(x)$ *does*) *by replacing in the double* nir*-integral* (4.27) *the polynomials* $\beta_k^*(\tau)$ *by the Laurent-type series*:

$$\beta_k^*(\tau) := \sum_{s=-1}^{+\infty} \beta_k\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} = \frac{\tau^{k+1}}{k+1} + \sum_{s=1}^{+\infty} (\ldots). \tag{4.66}$$

As usual, this applies both to the standard and non-standard[43] *choices of* β.

We may also take advantage of the identity $f^{\Uparrow\beta} := \beta(\partial_\tau) f(\frac{\tau}{n})$ to *formally* extend the *nir*-transform to functions f derived from an F with a zero//pole of order p at $x=0$:

$$F(x) = e^{-f} = x^p\, e^{-w(x)} \quad \text{with} \quad p \in \mathbb{Z}^* \quad \text{with} \quad w(.) \text{ regular at } 0.$$

That formal extension would read:

$$\begin{aligned} h(\nu) &\overset{\text{formally}}{=} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\frac{dn}{n}\int_0^\infty n^{-p\,\tau} \exp\left(p\lambda(\tau) - \beta(\partial_\tau) w\left(\frac{\tau}{n}\right)\right) d\tau \\ \tilde{\lambda}(\tau) &= \beta(\partial_\tau)(\log\tau) = \tau\log\tau - \tau + \sum_{0\leq s} \beta_{2s+1}\,(2s)!\,\tau^{-2s-1} \end{aligned}$$

[43] For non-standard choices, the series $\beta(\tau) := \sum \beta_s\, \tau^s$ has to be convergent if *nir* is to preserve convergence.

with λ denoting the Borel-Laplace resummation of the divergent series λ. However, in the above formula for $h(\nu)$, the first integration (in τ) makes no sense at infinity[44] and one would have to exchange the order of integration (first n, then τ), among other things, to make sense of the formula and arrive at the correct result, namely that the *nir*-transform turns functions of the form $f(x) = p \log x + Reg$ into Reg.[45] In other words, *there is no inner generator attached to the corresponding base point* $x = 0$. But it would be difficult to turn the argument into a rigorous proof, and so the best approaches remain the ones just outlined in the preceding subsection.

Directly extending *nir* to even more general test functions f would be possible, but increasingly difficult and of doubtful advantage. Extending *mir*, on the other hand, poses no difficulties.

Lemma 4.15 (Extending *mir*'s domain). *The* mir *transform* $\cancel{g} \to \hbar$ *extends, formally* and *analytically, to general transserial inputs* $\cancel{g}$ *of infinitesimal type, i.e.* $\cancel{g}(y) = o(y)$, $y \sim 0$, *and even to those with moderate growth* $\cancel{g}(y) = O(y^{-\sigma})$, $\sigma > 0$.

We face a similar situation when investigating the behaviour of $h(\nu)$ *over* $\nu = \infty$[46] for inputs f of the form:

$$f(x) = \text{polynomial}(x) \quad \text{or} \quad \text{polynomial}(x) + \sum_{i=1}^{i=N} p_i \log(x - x_i) \quad (p_i \in doZ^*).$$

Then the (4.40) expansion for $h(\nu)$ still converges in some suitable (ramified) neighbourhood of ∞ to some analytic germ, but the latter is no longer described by a power series (or a Laurent series, as we might expect a infinity) nor even by a (well-ordered) transseries, but by a complex combination both kinds of infinitesimals: small and large.[47]

Lemma 4.16 (Inverting *mir*). *The* mir *transform admits a formal inverse* $\text{mir}^{-1} : \hbar \to \cancel{g}$ *that acts, not just formally but also analytically, on general transserial inputs* $\hbar$ *of infinitesimal type. Like* mir, *this inverse*

[44] Even when interpreted term-by-term, *i.e.* after expanding $\exp\!\left(-\beta(\partial_\tau)(w(\frac{\tau}{n})\right)$.

[45] As long as $p \in \mathbb{Z}$.

[46] Under the change $\zeta = e^{\nu}$, this behaviour at $-\infty$ in the ν-plane translates into the behaviour over 0 in the ζ-plane.

[47] When re-interprented as a germ over 0 in the ζ-plane, it typically produces an essential singularity there, with Stokes phenomena and exponential growth or decrease, depending on the sector.

mir^{-1} *admits well-defined integro-differential components* $ID_{r,s}$ *of degree* r *and order* s*, but these are no longer of the form* $\partial^{-s} D_{r,s}$ *with a neat separation of the differentiations (coming first) and integrations* (*coming last*).

4.10 Parity relations

With the standard choice for β, we have the following parity relations for the *nir*-transform:

$$
\begin{aligned}
&F^{\vdash}(x) := 1/F(-x), \quad f^{\vdash}(x) := -f(-x) && \Longrightarrow \\
& && (\text{tangency } \kappa = 0)\\
&\mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nir}(f)(\nu) && (\text{tangency } \kappa = 0)\\
&\mathrm{nir}(f^{\vdash}) \text{ and } \mathrm{nir}(f) \text{ unrelated} && (\text{tangency } \kappa \text{ even} \geq 2)\\
&\mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nir}(f)(\epsilon_\kappa \nu) \text{ with } \epsilon_\kappa^{\frac{1}{\kappa+1}} = -1 && (\text{tangency } \kappa \text{ odd } \geq 1)\\
&\Rightarrow h^{\vdash}_{\frac{k}{\kappa+1}} = (-1)^{k-1} h_{\frac{k}{\kappa+1}} \text{ with} : (f, f^{\vdash}) \overset{\mathrm{nir}}{\mapsto} (h, h^{\vdash}) && \\
& && (\text{tangency } \kappa \text{ odd } \geq 1).
\end{aligned}
$$

For the *mir*-transform the parity relation doesn't depend on κ and assumes the elementary form :

$$\mathrm{mir}(-\cancel{g}) = -\mathrm{mir}(\cancel{g}).$$

5 Outer generators

5.1 Some heuristics

In the heuristical excursus at the beginning of the preceding section, we had chosen the driving function f such as to make the nearest singularity an *inner generator*. We must now hone f to ensure that the nearest singularity be an *outer generator*. For maximal simplicity, let us assume that:

$$0 \leq f(0) \leq +\infty \quad \text{and} \quad 0 < f(x) < +\infty \quad \text{for} \quad 0 < x \leq 1. \quad (5.1)$$

Thus $f^*(x) := \int_0^x f(x')dx'$ will be > 0 on the whole interval $]0, 1]$. Since we insist, as usual, on $F := \exp(-f)$ being meromorphic (5.1) leaves but three possibilities:

$$\begin{aligned}
&\textit{Case } 1: \quad 0 = F(0); && f(x) = -p\log(x) + \sum_{k=0}^{k=\infty} f_k x^k \\
&\textit{Case } 2: \quad 0 < F(0) < 1; && f(x) = \sum_{k=0}^{k=\infty} f_k x^k \quad (f_0 > 0) \\
&\textit{Case } 3: \quad F(0) = 1; && f(x) = \sum_{k=\kappa}^{k=\infty} f_k x^k \quad (f_\kappa > 0,\ \kappa \geq 1).
\end{aligned}$$

In all three cases, the nearest singularity of $j(\zeta)$ (*cf.* (1.1)) is located at $\zeta = 1$ and reflects the n-asymptotics of the Taylor coefficients $J(n)$.

$$J(n) := \sum_{m=\epsilon}^{m=n-1} \prod_{k=\epsilon}^{k=m-1} F\left(\frac{k}{n}\right) \ \big(\epsilon \in \{0,1\}\big) \Longrightarrow \tilde{J}(n) := \sum_{k \geq 0} j_k\, n^{-k}. \quad (5.2)$$

Case 1. Is simplest.[48] By truncating the $\sum\prod$ expansion at $m = m_0$, we get the exact values of all coefficients j_k up to $k = p\, m_0$.

Case 2. Corresponds to tangency 0. Here, finite truncations yield only approximate values. To get the exact coefficients, we must harness the full $\sum\prod$ expansion but we still end up with closed expressions for each j_k.

Case 3. Corresponds to tangency $\kappa \geq 1$ case. Here, again, the full $\sum\prod$ expansion must be taken into account, but the difference is that we now get coefficients j_k which, though exact, are no longer neatly expressible in terms of elementary functions.

5.2 The short and long chains behind *nur/mur*

Let us now translate the above heuristics into precise (non-linear) functionals. For Case 1, the definition is straightforward:

The short, four-link chain:

$$F \xrightarrow{1} k \xrightarrow{2} h \xrightarrow{3} h' \xrightarrow{4} H \qquad h = \overset{\frown}{\ell u},\ h' = \overset{\wedge}{\ell u},\ H = \overset{\wedge}{L u}\ .$$

[48] Here, we must take $\epsilon = 0$ to avoid an all-zero result.

Details of the four steps:

$$F(x) := F_1\, x + F_2\, x^2 + F_3\, x^3 + \ldots \qquad \text{(converg}^t\text{)}$$

$$\downarrow^1$$

$$k(n) := \Big(F\big(\tfrac{1}{n}\big) + F\big(\tfrac{1}{n}\big)F\big(\tfrac{2}{n}\big) + F\big(\tfrac{1}{n}\big)F\big(\tfrac{2}{n}\big)F\big(\tfrac{3}{n}\big) + \ldots\Big)/Ig_F(n) \ \text{(diverg}^t\text{)}$$

$$\|$$

$$k(n) := \sum_{1\le s} k_s\, \frac{1}{n^k} \qquad \text{(N.B. } k \equiv J^{\#} \text{ as in (1.3))} \qquad \text{(diverg}^t\text{)}$$

$$\downarrow^2$$

$$h(\nu) := \sum_{1\le s} k_s\, \frac{\nu^s}{s!} \qquad \text{(converg}^t\text{)}$$

$$\downarrow^3$$

$$h'(\nu) := \sum_{1\le s} k_s\, \frac{\nu^{s-1}}{(s-1)!} \qquad \text{(converg}^t\text{)}$$

$$\downarrow^4$$

$$H(\zeta) := h'(\log(1+\zeta)).$$

Mark the effect of removing the ingress factor Ig_F after the first step. If

$$F(x) = c_0\, x^d\, F_*(x) \text{ with } F_*(x) = 1 + \cdots \in \mathbb{C}\{x^2\} \text{ (respectively } \mathbb{C}\{x\})$$

then, according to the results of Section 3, removing Ig_F amounts to dividing $k(n)$ by $c_0^{-1/2}(2\pi n)^{d/2}$ and integrating $d/2$ times the functions[49] $h(\nu)$ or $h'(\nu)$. The removal of the ingress factor thus has three main effects:

(i) as already pointed out, it makes the outer generators independent of the ingress point;[50]
(ii) depending on the sign of d, it renders the singularities smoother (for $d > 0$) or less smooth (for $d < 0$), in the ν- or ζ-planes alike;
(iii) depending on the parity of d, it leads in the Taylor expansions of the minors $\overset{\frown}{\ell u}\ (\nu) := h(\nu)$ and $\overset{\wedge}{\ell u}\ (\nu) := h'(\nu)$ either to integral powers of ν (for d even) or to strictly semi-integral powers (for d odd). This means that the corresponding majors $\overset{\smile}{\ell u}$ and $\overset{\vee}{\ell u}$ and, by way of

[49] Or more accurately $c_0^{1/2}(2\pi)^{-d/2}h(\nu)$ and $c_0^{1/2}(2\pi)^{-d/2}h'(\nu)$.

[50] Just as was the case with the inner generators.

consequence, the inner generators themselves, will carry *logarithmic* singularities (for d even) or strictly semi-integral powers (for d odd).[51]

Time now to deal with the Cases 2 and 3 (*i.e.* $F(0) \neq 0$). These cases lead to a nine-link chain quite similar to that which in Section 4.2 did service for the *inner generators*, but with the key steps *nir* and *mir* significantly altered into *nur* and *mur*:

The long, nine-link chain:

$$
\begin{array}{cccccccc}
 & \xrightarrow{\mathbf{nur}} & \rightarrow & \rightarrow & \rightarrow & \xrightarrow{\mathbf{nur}} & H & \qquad H = \widehat{Lu} \\
\uparrow & & & & & \downarrow & \uparrow^{9} & \\
\uparrow & & f^* & \xrightarrow{3} & g^* & \downarrow & h' & \qquad h' = \widehat{lu} \\
\uparrow & & \uparrow^{2} & & \downarrow^{4} & \downarrow & \uparrow^{8} & \\
 & \xleftarrow{\mathbf{nur}} & f & & g & \xrightarrow{\mathbf{nur}} & h & \qquad h = \overset{\frown}{lu} \\
 & & \uparrow^{1} & & \downarrow^{5} & & \uparrow^{7} & \\
 & & F & & \cancel{g} & \underset{\mathbf{mur}}{\xrightarrow{6}} & \hbar &
\end{array}
$$

Details of the nine steps:

$\xrightarrow{1}$:	precomposition	: $F \to f$	with	$f(x) := -\log F(x)$
$\xrightarrow{2}$:	integration	: $f \to f^*$	with	$f^*(x) := \int_0^x f(x_0)\, dx_0$
$\xrightarrow{3}$:	reciprocation	: $f^* \to g^*$	with	$f^* \circ g^* = id$
$\xrightarrow{4}$:	derivation	: $g^* \to g$	with	$g(y) := \frac{d}{dy} g^*(y)$
$\xrightarrow{5}$:	inversion	: $g \to \cancel{g}$	with	$\cancel{g}(y) := 1/g(y)$
$\xrightarrow{6}$:	**mur** functional	: $\cancel{g} \to \hbar$		see Section 5.4 *infra*
$\xrightarrow{7}$:	inversion	: $\hbar \to h$	with	$h(\nu) := 1/\hbar(\nu)$
$\xrightarrow{8}$:	derivation	: $h \to h'$		$h'(\nu) := \frac{d}{d\nu} h(\nu)$
$\xrightarrow{9}$:	postcomposition	: $h' \to H$	with	$H(\zeta) := h'(\log(1+\zeta)$
$\xrightarrow{2...7}$:	**nur** functional	: $g \to h$	with	see Section 5.3 *infra*.

[51] Removing the ingress factor has exactly the opposite effect on *inner generators*: these generically carry semi-integral powers for d even and logarithmic singularities for d odd.

5.3 The *nur* transform

Integral-serial expression of *nur*

Starting from f we define $f^{\Uparrow\mathfrak{b}^*}$, $f^{\uparrow\mathfrak{b}^*}$ and $f^{\Uparrow\beta^*}$, $f^{\uparrow\beta^*}$ as follows:

$$f(x)=\sum_{k\geq\kappa} f_k\, x^k \qquad (\kappa\geq 1,\ f_\kappa\neq 0)$$

$$f^{\Uparrow\mathfrak{b}^*}(n,\tau):=\mathfrak{b}(\partial_\tau)f\left(\frac{\tau}{n}\right)=\sum_{k\geq\kappa} f_k\, n^{-k}\,\mathfrak{b}_k^*(\tau)=:f_\kappa\frac{n^{-\kappa}\tau^{\kappa+1}}{\kappa+1}+f^{\uparrow\mathfrak{b}^*}(n,\tau)$$

$$f^{\Uparrow\beta^*}(n,\tau):=\beta(\partial_\tau)f\left(\frac{\tau}{n}\right)=\sum_{k\geq\kappa} f_k\, n^{-k}\,\beta_k^*(\tau)=:f_\kappa\frac{n^{-\kappa}\tau^{\kappa+1}}{\kappa+1}+f^{\uparrow\beta^*}(n,\tau)$$

with the usual definitions in the standard case:

$$\mathfrak{b}_k^*(\tau):=\frac{B_{k+1}(\tau+1)}{k+1}=\beta_k^*(\tau+\frac{1}{2})$$

$$\beta_k^*(\tau):=\frac{B_{k+1}(\tau+\frac{1}{2})}{k+1}$$

where B_k stands for the k^{th} Bernoulli polynomial. Recall that $\mathfrak{b}_k^*(m)$ is a polynomial in m of degree $k+1$, with leading term $\frac{m^{k+1}}{k+1}$. Then the *nur*-transform is defined as follows:

$$nur : f \mapsto h \tag{5.3}$$

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(n\,\nu)\,\frac{dn}{n}\sum_{m=0}^{\infty}\exp^{\#}(-f^{\Uparrow\mathfrak{b}^*}(n,m)) \tag{5.4}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(n\,\nu)\frac{dn}{n}\sum_{m=0}^{\infty}\exp\left(-f_\kappa\frac{n^{-\kappa}m^{\kappa+1}}{\kappa+1}\right)\exp_{\#}\big(-f^{\uparrow\mathfrak{b}^*}(n,m)\big)$$

or equivalently (and preferably):

$$h(\nu)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(n\,\nu)\frac{dn}{n}\sum_{m\in\frac{1}{2}+\mathbb{N}}^{\infty}\exp^{\#}(-f^{\Uparrow\beta^*}(n,m)) \tag{5.5}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(n\,\nu)\frac{dn}{n}\sum_{m\in\frac{1}{2}+\mathbb{N}}^{\infty}\exp\left(-f_\kappa\frac{n^{-\kappa}m^{\kappa+1}}{\kappa+1}\right)\exp_{\#}\big(-f^{\uparrow\beta^*}(n,m)\big)$$

where $\exp_{\#}(X)$ denotes the usual exponential function, but expanded as a power series of X. Similarly, $\exp^{\#}(X)$ denotes the exponential expanded as a power series of X *minus* the leading term of X, which remains within the exponential. An unmarked $\exp(X)$, on the other hand, should be construed as the usual exponential function.

The analytical expressions vary depending on the tangency order κ. Indeed, after expanding $\exp_{\#}(\ldots)$, we are left with the task of calculating individual sums of type:

$$S_{0,k}(f_0) = \sum_{m\in\mathbb{N}} m^k \exp(-f_0\, m) \qquad \text{in Case 2} \quad (\kappa = 0,\ f_0 > 0)$$

$$S_{\kappa,k}(f_\kappa) = \sum_{m\in\mathbb{N}} m^k \exp\left(-f_\kappa \frac{n^{-\kappa}\, m^{\kappa+1}}{\kappa+1}\right) \text{ in Case 3} \quad (\kappa \geq 1,\ f_\kappa > 0)$$

or of type:

$$Z_{0,k}(f_0) = \sum_{m\in\frac{1}{2}+\mathbb{N}} m^k \exp(-f_0 m) \qquad \text{in Case 2} \quad (\kappa = 0,\ f_0 > 0)$$

$$Z_{\kappa,k}(f_\kappa) = \sum_{m\in\frac{1}{2}+\mathbb{N}} m^k \exp\left(-f_\kappa \frac{n^{-\kappa} m^{\kappa+1}}{\kappa+1}\right) \text{ in Case 3} \quad (\kappa \geq 1,\ f_\kappa > 0).$$

Since we assumed f_κ to be positive in all cases, convergence is immediate and precise bounds are readily found. However, only for $\kappa = 0$ do the sums $S_{\kappa,k}$, $Z_{\kappa,k}$ admit closed expressions for all k. For the former sums we get:

$$S_{0,k}(\alpha) = \frac{L_k(a)}{(1-a)^{k+1}} \quad \text{with} \quad a := e^{-\alpha};$$

$$L_k(a) := \mathrm{tr}_k\left((1-a)^{k+1} \sum_{0\leq s\leq 2k} s^k\, a^s\right)$$

where tr_k means that we truncate after the k^{th} power of a, which leads to self-symmetrical polynomials of the form:

$$L_k(a) = a + (1+k+2^k)\, a^2 + \cdots + (1+k+2^k)\, a^{k-1} + a^k \quad \textit{with} \quad L_k(1) = k!$$

For the latter sums we get the generating function:

$$\sum_{0\leq k} Z_{0,k}(\alpha) \frac{\sigma^k}{k!} = \frac{1}{e^{\frac{1}{2}(\alpha-\sigma)} - e^{-\frac{1}{2}(\alpha-\sigma)}} = \frac{1}{\alpha-\sigma} + \text{Regular}(\alpha-\sigma). \quad (5.6)$$

Hence:

$$Z_{0,k}(\alpha) = \frac{k!}{\alpha^{k+1}} + \text{Regular}(\alpha). \qquad (5.7)$$

Let us now justify the above definition of *nur*. For a tangency order $\kappa \geq 0$ and a driving function $f(x) := \sum_{s\geq\kappa} f_s\, x^s$ as in the cases 2 or 3 of Section 5.1, our Taylor coefficients $J(n)$ will have the following asymptotic expansions, *before* and *after* division by the ingress factor $Ig_F(n)$:

$$\tilde{J}(n) := \sum_{0\leq m} \exp\left(- \sum_{0\leq k\leq m} f\left(\frac{k}{m}\right)\right) \tag{5.8}$$

$$= \sum_{0\leq m} \exp\left(-(m+1)\, f_0 - \sum_{1\leq s} n^{-s} \left(\mathfrak{b}_s^*(m) - \mathfrak{b}_s^*(0)\right) f_s \right) \tag{5.9}$$

$$\tilde{Ig}_F(n) := \exp\left(-\frac{1}{2}\, f_0 + \sum_{1\leq s} n^{-s}\, \mathfrak{b}_s^*(0)\, f_s\right) \tag{5.10}$$

$$\tilde{J}(n)/\tilde{Ig}_F(n) = \sum_{0\leq m} \exp\left(- \left(m + \frac{1}{2}\right) f_0 - \sum_{1\leq s} n^{-s}\, \mathfrak{b}_s^*(m)\, f_s\right) \tag{5.11}$$

$$= \sum_{0\leq m} \exp\left(- \sum_{0\leq s} n^{-s}\, \mathfrak{b}_s^*(m)\, f_s. \right) \tag{5.12}$$

Of course, the summand $\frac{1}{2} f_0$ automatically disappears when the tangency order κ is > 0. But, whatever the value of κ, the hypothesis $f_\kappa > 0$ ensures the convergence of the m-summation[52] in (5.12), which yields, in front of any given power n^{-s}, a well-defined, finite coefficient. If we then suject the right-hand side of (5.12), term-wise, to the (upper) Borel transform $n \to \nu$, we are led straightaway to the above definition of the *nur*-transform $f(x) \mapsto h(\nu)$.

5.4 Expressing *nur* in terms of *nir*

Lemma 5.1 (Decomposition of ***nur*****).** *The nur-transforms reduces to an alternating sum of nir-transforms*:

$$\text{nur}(f) = \sum_{p\in\mathbb{Z}} (-1)^p\, \text{nir}(2\pi i\; p + f). \tag{5.13}$$

[52] After factoring $\exp(-\sum_{\kappa\leq s}(\ldots))$ into $\exp(-\sum_{\kappa=s}(\ldots))\, \exp_\#(-\sum_{\kappa<s}(\ldots))$ and expanding the second factor as a power series of $(\sum_{\kappa<s}(\ldots))$.

It suffices to show that this holds term-by-term, *i.e.* for the coefficient of each monomial ν^n on the left- and right-hand sides of (5.13). For $\kappa = 0$ for instance, this results from the identities:

$$\sum_{m\in\frac{1}{2}+\mathbb{Z}} m^k \exp(-f_0\, m) = \sum_{p\in\mathbb{Z}} (-1)^p \frac{(k+1)!}{(2\pi i\, p + f_0)^{k+1}} \tag{5.14}$$

which are a direct consequence of Poisson's summation formula.[53] The same argument applies for $\kappa > 0$.

As a consequence of the above lemma, we see that whereas the *nir*-transform depends on the exact determination of $log F$, the *nur*-transform depends only on the determination of $F^{1/2}$. This was quite predictable, in view of the interpretation of *nur*.[54].

5.5 The *mur* transform

Since in this new nine-link chain (of Section 5.2) all the steps but *mur* are elementary, and the composite step *nur* has just been defined, that indirectly determines *mur* itself, just as knowing *nir* determined *mir* in the preceding section. There are, however, two basic differences between *mur* and *mir*.

(i) *Analytic difference*: whereas the singularities of a *mir*-transform were *mir*-transforms of singularities (reflecting the essential closure of the inner algebra), the singularities of *mur*-transforms are *mir*-transforms (not *mur*-transforms!) of singularities (reflecting the non-recurrence of outer generators under alien derivation).
(ii) *Formal difference*: unlike *mir*, *mur* doesn't reduce to a purely integro-differential functional. It does admit interesting, if complex, expressions[55] but we needn't bother with them, since the whole point of deriving an exact analytical expression for *mir* was to account for the closure phenomenon just mentioned in (i) but which no longer applies to *mur*.

---

[53] Decompose the left-hand side of (5.14) as $\sum_{m\in\frac{1}{2}+\mathbb{Z}} = \sum_{m\in\frac{1}{2}\mathbb{Z}} - \sum_{m\in\mathbb{Z}}$ and formally apply Poisson's formula separately to each sum.

[54] The square root of F comes from our having replaced j_F by $j_F^{\#}$, *i.e.* from dividing by the ingress factor, which carries the term $e^{-f_0/2} = F(0)^{1/2}$.

[55] Somewhat similar to the expression for the generalised (non-standard) *mir*-transform when we drop the condition $\beta_0 \neq 0$.

5.6 Translocation of the *nur* transform

Like with *nir*, it is natural to "translocate" *nur*, *i.e.* to measure its failure to commute with translations. To do this, we have the choice, once again, between four expressions (where $\eta := \int_0^\epsilon f(x)\,dx$)

$$\begin{aligned}
&\text{choice } 1: \ (\text{nur}\ e^{\epsilon\partial_x} - e^{\eta\partial_\nu}\ \text{nur})\ f \ \text{ as a function of } (\epsilon, f)\\
&\text{choice } 2: \ (\text{nur}\ e^{\epsilon\partial_x} - e^{\eta\partial_\nu}\ \text{nur})\ f \ \text{ as a function of } (\eta, f)\\
&\text{choice } 3: \ (\text{nur} - e^{-\eta\partial_\nu}\ \text{nur}\ e^{\epsilon\partial_x})\ f \ \text{ as a function of } (\epsilon, f)\\
&\text{choice } 4: \ (\text{nur} - e^{-\eta\partial_\nu}\ \text{nur}\ e^{\epsilon\partial_x})\ f \ \text{ as a function of } (\eta, f)
\end{aligned}$$

but whichever choice we make (let us think of choice 3, for consistency) two basic differences emerge between *nir*'s and *nur*'s translocations:

(i) *Analytic difference*: the finite or infinitesimal increments $\nabla h(\epsilon, \nu)$ or $\delta h_m(\nu)$ defined as in Section 4.6 but with respect to *nur*, are no longer entire functions of their arguments, even when the driving function f is entire or polynomial. The reason for this is quite simple: with the *nir*-transform, to a shift ϵ in the x-plane there answers a well-defined shift $\eta = \int_0^\infty f(x)dx$ in the ν-plane, calculated from a well-defined determination of $f = -\log F$, but this no longer holds with the *nur*-transform, whose construction involves *all* determinations of f;

(ii) *Formal difference*: these increments still admit exact analytical expansions somewhat similar to (4.48) and (4.58) but the formulas are now more complex[56] and above all less useful. Indeed, the main point of these formulas in the *nir* version was to establish that the increments $\nabla h(\epsilon, \nu)$ or $\delta h_m(\nu)$ were entire functions of ϵ and ν, but with *nur* this is no longer the case, as was just pointed out.

5.7 Removal of the ingress factor

As we saw, changing j_F into $j_F^{\#}$ brings rather different changes to the construction of the *inner* and *outer* generators: for the *inner* generators it means merging the critical stationary factor J_4 with the *egress* factor

[56] With twisted equivalents of the convolution (4.54), under replacement of the factorials by q-factorials.

Eg_F; for the *outer* generators it means pruning the critical stationary factor J of the *ingress* factor Ig_F. Nonetheless, the end effect is exactly the same: the parasitical summands $\mathfrak{b}_s^*(0)$ vanish from (4.12) and (5.12) alike.

5.8 Parity relations

$$F^{\vdash}(x) := 1/F(-x), \quad f^{\vdash}(x) := -f(-x) \qquad \Longrightarrow$$
$$\mathrm{nur}(f^{\vdash})(\nu) = -\mathrm{nur}(f)(\nu) \qquad\qquad (\text{tangency } \kappa = 0).$$

6 Inner generators and ordinary differential equations

In some important instances, namely for *all* polynomial inputs f and *some* rational inputs F, the corresponding inner generators happen to verify ordinary differential equation of a rather simple type – *linear homogeneous with polynomial coefficients* – but often of high degree. These ODEs are interesting on three accounts:

(i) they lead to an alternative, more classical derivation of the properties of these inner generators;
(ii) they yield a precise description of their behaviour *over* ∞ in the ν-plane, *i.e. over* 0 in the ζ-plane;
(iii) they stand out, among similar-looking ODEs, as leading to a *rigid* resurgence pattern, with essentially *discrete* Stokes constants, insensitive to the *continuously* varying parameters.

6.1 "Variable" and "covariant" differential equations

As usual, we consider four types of shift operators $\beta(\partial_\tau)$, relative to the choices

$$\text{trivial choice} \qquad \beta(\tau) := \tau^{-1} \tag{6.1}$$
$$\text{standard choice} \qquad \beta(\tau) := \left(e^{\tau/2} - e^{-\tau/2}\right)^{-1} = \tau^{-1} - \frac{1}{24}\tau + \dots \tag{6.2}$$
$$\text{odd choice} \qquad \beta(\tau) := \tau^{-1} + \sum_{s\geq 0} \beta_{2s+1}\, \tau^{2s+1} \tag{6.3}$$
$$\text{general choice} \qquad \beta(\tau) := \tau^{-1} + \sum_{s\geq 0} \beta_s\, \tau^s . \tag{6.4}$$

We then apply the *nir*-transform to a driving function f such that $f(0) = 0$, with special emphasis on the case $f'(0) \neq 0$:

$$f(x) := \sum_{1 \leq s \leq r} f_s\, x^s \tag{6.5}$$

$$\varphi(n,\tau) := \beta(\tau) f\left(\frac{\tau}{n}\right) = \frac{1}{2}\frac{\tau^2}{n} f_1 + \ldots \quad \in \mathbb{C}[n^{-1}, \tau] \tag{6.6}$$

$$\varphi(n,\tau) := \varphi^+(n,\tau) + \varphi^-(n,\tau) \quad \text{with} \quad \varphi^\pm(n, \pm\tau) \equiv \pm\varphi^\pm(n,\tau) \tag{6.7}$$

$$k(n) := \left[\int_0^\infty \exp^{\#}(\varphi(n,\tau))\, d\tau\right]_{\text{singular}} \tag{6.8}$$

$$:= \int_0^\infty \exp^{\#}(\varphi(n,\tau)) \cosh_{\#}(\varphi^-(n,\tau))\, d\tau \quad (\text{if} \ \ f_1 \neq 0) \tag{6.9}$$

$$\hat{k}(\nu) := \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\, e^{\nu\, n}\, dn\right]_{\text{formal}} = h'(\nu) \tag{6.10}$$

$$\overset{\frown}{k}(\nu) := \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\, e^{\nu\, n}\, \frac{dn}{n}\right]_{\text{formal}} = h(\nu). \tag{6.11}$$

But the case $f(0) \neq 0$ also matters, because it corresponds the so-called "exceptional" or "movable" generators. In that case the *nir*-transform produces no fractional powers. So we set:

$$f(x) := \sum_{0 \leq s \leq r} f_s\, x^s \tag{6.12}$$

$$k^{\text{total}}(n) := \int_0^\infty \exp^{\#}(\varphi(n,\tau))\, d\tau \tag{6.13}$$

$$\hat{k}{}^{\text{total}}(\nu) := \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k^{\text{total}}(n)\, e^{\nu\, n}\, dn\right]_{\text{formal}} \tag{6.14}$$

$$\overset{\frown}{k}{}^{\text{total}}(\nu) := \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k^{\text{total}}(n)\, e^{\nu\, n}\, \frac{dn}{n}\right]_{\text{formal}} \tag{6.15}$$

The above definitions also extend to the case $f(0) = 0$. The *nir*-transform then produces a mixture of entire and fractional powers, and the index *total* affixed to k signals that we take them all.

For polynomial inputs, both k^{total} and k along with their Borel transforms verify remarkable linear-homogeneous ODEs. The ones verified by k^{total} are dubbed *variable* because there is no simple description of how they change when the base point changes in the x-plane (*i.e.* when the driving function undergoes a shift from f to ${}^{\epsilon}f$). The ODEs verified by k, on the other hand, deserve to be called *covariant*, for two reasons:

(i) when going from a proper base-point x_i to another proper base-point x_j (*proper* means that $f(x_i) = 0,\ f(x_j) = 0$), these covariant ODEs verified by $\hat{k}\,(\nu)$ simply undergo a shift $\nu = \int_{x_i}^{x_j} f(x)dx$ in the ν-plane;
(ii) there is a unique extension of the covariant ODE even to non-proper base-points x_i (*i.e.* when $f(x_i) \neq 0$), under the same formal covariance relation as above. That extension, of course, doesn't coincide with the *variable* ODE.[57]

"Variable" and "covariant" linear-homogeneous polynomial ODEs
They are of the form:

$$\text{variable ODE:}\quad P_v(n,-\partial_n)k^{\text{total}}(n)=0 \Leftrightarrow P_v(\partial_\nu,\nu)\,\widehat{k}^{\text{total}}(\nu)=0 \tag{6.16}$$

$$\text{covariant ODE:}\ P_c(n,-\partial_n)k(n)=0 \qquad \Leftrightarrow P_c(\partial_\nu,\nu)\,\hat{k}\,(\nu)=0 \tag{6.17}$$

with polynomials

$$P_v(n,\,\nu) = \sum_{0\le p\le d}\ \sum_{0\le q\le \delta} \mathrm{dv}_{\mathrm{p,q}}\ \mathrm{n^p}\ \nu^{\mathrm{q}} \tag{6.18}$$

$$P_c(n,\,\nu) = \sum_{0\le p\le d}\ \sum_{0\le q\le \delta} \mathrm{dc}_{\mathrm{p,q}}\ \mathrm{n^p}\ \nu^{\mathrm{q}} \tag{6.19}$$

of degree d and δ in the non-commuting variables n and ν: $\quad [n,\,\nu] = 1$. The covariance relation reads:

$$\begin{aligned} &P_c^{\,{}^{\epsilon}f}(n,\,\nu - \eta) \equiv P_c^{\,f}(n,\,\nu)\ \forall\epsilon \quad \text{with} \quad {}^{\epsilon}f(x) = f(x+\epsilon) \\ &\qquad \text{and} \quad \eta := \int_0^{\epsilon} f(x)dx \end{aligned} \tag{6.20}$$

[57] For a proper base-point, on the other hand, the variable ODEs, though still distinct from the covariant ones, are *also* verified by k.

Existence and calculation of the variable ODEs
For any $s \in \mathbb{N}$ let φ_s, ψ_s denote the polynomials in (n^{-1}, τ) characterised by the identities:

$$\partial_n^s\, k^{\text{total}}(n) = \int_0^0 \varphi_s(n,\tau)\, \exp^{\#}(\varphi(n,\tau))\, d\tau \tag{6.21}$$

$$\text{with } \varphi_s(n,\tau) \in \mathbb{C}[\partial_n\varphi, \partial_n^2\varphi, \ldots, \partial_n^s\varphi] \in \mathbb{C}[n^{-1},\tau] \tag{6.22}$$

$$\int_0^\infty d_\tau^s\big(\tau^k \exp^{\#}(\varphi(n,\tau))\big) = \int_0^\infty \psi_s(n,\tau)\exp^{\#}(\varphi(n,\tau))d\tau = 0 \tag{6.23}$$

$$\text{with } \psi_s(n,\tau) = \tau^s\partial_\tau\varphi(n,\tau) + s\tau^{s-1} \in \mathbb{C}[n^{-1},\tau]. \tag{6.24}$$

For δ, δ' large enough, the first polynomials $\{\varphi_s\,;\ s \le \delta\}$ and $\{\psi_s\,;\ s \le \delta'\}$ become linearly dependent on $\mathbb{C}[n^{-1}]$ or, what amounts to the same, on $\mathbb{C}[n]$. So we have relations of the form:

$$\begin{aligned} 0 &= \sum_{0\le s\le\delta} A_s(n)\,\varphi_s(n,\tau) + \sum_{0\le s\le\delta'} B_s(n)\,\psi_s(n,\tau) \\ &\text{with} \quad A(n), B(n) \in \mathbb{C}[n] \end{aligned} \tag{6.25}$$

and to each such relation there corresponds a linear ODE for k^{total}:

$$\left(\sum_{0\le s\le\delta} A_s(n)\,\partial_n^s\right) k^{\text{total}}(n) = 0. \tag{6.26}$$

Existence and calculation of the covariant ODEs for $f(0) = 0$
For each $s \in \mathbb{N}$ let $\varphi_s^{\pm}$ and $\psi_s^{\pm\pm}$, $\psi_s^{\pm\mp}$ denote the polynomials in (n^{-1}, τ) characterised by the identities:

$$\begin{aligned} &\partial_n^s\, k(n) \\ &= \int_0^\infty \Big(\varphi_s^+(n,\tau)\cosh(\varphi^-(n,\tau)) + \varphi_s^-(n,\tau)\sinh(\varphi^-(n,\tau))\Big)\, e^{\varphi^+(n,\tau)}\, d\tau \end{aligned}$$

$$\text{with } \varphi_s^{\pm}(n,\tau) \in \mathbb{C}[\partial_n\varphi^+, \ldots, \partial_n^s\varphi^+, \partial_n\varphi^-, \ldots, \partial_n^s\varphi^-] \in \mathbb{C}[n^{-1},\tau]$$

$$\int_0^\infty d_\tau^s\big(\tau^k e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau))\big) = \int_0^\infty d_\tau^s\big(\tau^k e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau))\big) = 0$$

with

$$\begin{aligned} d_\tau^s\big(\tau^k e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau))\big) &= +\varphi_s^{++}(n,\tau)e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau)d\tau \\ &\quad + \varphi_s^{+-}(n,\tau)e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau)d\tau \\ d_\tau^s\big(\tau^k e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau))\big) &= +\varphi_s^{--}(n,\tau)e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau)d\tau \\ &\quad + \varphi_s^{-+}(n,\tau)\, e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau)d\tau. \end{aligned}$$

Here again, for δ, δ' large enough, there are going to be dependence relations of the form:

$$
\begin{aligned}
0 = &\sum_{0 \le s \le \delta} A_s(n)\, \varphi^{+}(n,\tau) + \sum_{0 \le s \le \delta'} B_s(n)\, \psi^{++}(n,\tau) \\
&+ \sum_{0 \le s \le \delta'} C_s(n)\, \psi^{++}(n,\tau)
\end{aligned} \tag{6.27}
$$

$$
0 = \sum_{0 \le s \le \delta} A_s(n)\, \varphi^{-}(n,\tau) + \sum_{0 \le s \le \delta'} B_s(n)\, \psi^{+-}(n,\tau) \tag{6.28}
$$

$$
\begin{aligned}
&+ \sum_{0 \le s \le \delta'} C_s(n)\, \psi^{-+}(n,\tau) \\
&\text{with} \quad A(n), B(n), C(n) \in \mathbb{C}[n]
\end{aligned} \tag{6.29}
$$

and to each such relation there will corresponds a linear ODE for k:

$$
\left(\sum_{0 \le s \le \delta} A_s(n)\, \partial_n^s \right) k(n) = 0. \tag{6.30}
$$

Remark. Although the above construction applies, strictly speaking, only to the case of tangency $\kappa = 1$, *i.e.* to the case $f_0 = 0$, $f_1 \neq 0$, it is in fact universal. Indeed, if we set $f_0 = f_1 = \cdots = f_{\kappa-1}$, $f_\kappa \neq 0$ in the covariant ODEs thus found, we still get the correct covariant ODEs for a general tangency order $\kappa > 1$.

Existence and calculation of the covariant ODEs for $f(0) \neq 0$
There are five steps to follow:

(i) fix a degree r and calculate $P^f(n,\nu)$ by the above method for an arbitrary f of degree r such that $f(0) = 0$;
(ii) drop the assumption $f(0) = 0$ but subject f to a shift ϵ such that ${}^{\epsilon}f(0) = f(\epsilon) = 0$ and apply (i) to calculate $P^{{}^{\epsilon}f}(n,\nu)$ without actually solving the equation $f(\epsilon) = 0$ (keep ϵ as a free variable);
(iii) calculate the ϵ-polynomial $P^{{}^{\epsilon}f}(n,\nu - f^*(\epsilon))$ with $f^*(x) := \int_0^x f(t)dt$ as usual;
(iv) divide it by the ϵ-polynomial $f(\epsilon)$ (momentarily assumed to be $\neq 0$) and calculate the remainder P_0 and quotient P_1 of that division:

$$
P^{{}^{\epsilon}f}(n,\nu - f^*(\epsilon)) =: P_0^f(n,\nu,\epsilon) + P_1^f(n,\nu,\epsilon)\, f(\epsilon);
$$

(v) use the covariance identity $P^{\epsilon f}(n, \nu - f^*(\epsilon)) \equiv P^f(n, \nu)\ \forall\epsilon$ to show that the remainder $P_0^f(n, \nu, \epsilon)$ is actually constant in ϵ. Then set

$$P^f(n, \nu) := P_0^f(n, \nu, 0).$$

6.2 ODEs for polynomial inputs f. Main statements

Dimensions of spaces of variable ODEs

For $r := \deg(f)$ and for each pair $(x.y.)$ with

$$x \in \{v, c\} = \{\text{variable, covariant}\}$$
$$y \in \{t, s, o, g\} = \{\text{trivial, standard, odd, general}\}$$

the dimension of the corresponding space of ODEs is always of the form:

$$\dim_{\text{x.y.}}(r, d, \delta) \equiv \big(d - A_{x.y.}(r)\big)\big(\delta - B_{x.y.}(r)\big) - C_{x.y.}(r) \quad (6.31)$$

with δ (respectively d) denoting the differential order of the the ODEs in the n-variable (respectively in the ν-variable). Of special interest are the extremal pairs $(\underline{d}, \overline{\delta})$ and $(\overline{d}, \underline{\delta})$ with

$$\underline{d} = 1 + A_{x.y.}(r) \qquad \overline{\delta} = 1 + B_{x.y.}(r) + C_{x.y.}(r) \quad (6.32)$$

$$\overline{d} = 1 + A_{x.y.}(r) + C_{x.y.}(r) \qquad \underline{\delta} = 1 + B_{x.y.}(r) \quad (6.33)$$

($\underline{d}$ and $\underline{\delta}$ minimal; $\overline{d}$ and $\overline{\delta}$ co-minimal) because the corresponding dimension is exactly 1.

Dimensions of spaces of variable ODEs

$$\dim_{\text{v.t.}}(r, d, \delta) = (d-r)\,(\delta - r - 1) - \frac{1}{2}r^2 + \frac{1}{2}r - 1$$

$$\dim_{\text{v.s.}}(r, d, \delta) = (d-r)\,(\delta - r^2 - 2r + 1) - \frac{1}{2}r^2(r+1) \qquad (r \text{ even})$$

$$= (d-r)\,(\delta - r^2 - 2r) - \frac{1}{2}(r^3 + r^2 - 5r + 5) \quad (r \text{ odd})$$

$$\dim_{\text{v.o.}}(r, d, \delta) = (d-r)\,(\delta - r^2 - 2r + 1) - \frac{1}{2}r^2(r+1) \qquad (r \text{ even})$$

$$= (d-r)\,(\delta - r^2 - 2r) - \frac{1}{2}(r^3 + r^2 - 3r + 3) \quad (r \text{ odd} \neq 3)$$

$$\dim_{\text{v.g.}}(r, d, \delta) = (d-r)\,(\delta - r^2 - 2r) - \frac{1}{2}r^2(r+1).$$

Dimensions of spaces of covariant ODEs

$$\dim_{\text{c.t.}}(r,d,\delta)=(d-r+1)\,(\delta-r+1)-\frac{1}{2}(r-1)(r-2)$$

$$\dim_{\text{c.s.}}(r,d,\delta)=(d-r+1)\,(\delta-r^2-r+1)-\frac{1}{2}r^2(r-1)\qquad (r\ \text{even})$$

$$=(d-r+1)\,(\delta-r^2-r+1)-\frac{1}{2}(r^2-5)(r-1)\ (r\ \text{odd})$$

$$\dim_{\text{c.o.}}(r,d,\delta)=(d-r+1)\,(\delta-r^2-r+1)-\frac{1}{2}r^2(r-1)\qquad (r\ \text{even})$$

$$=(d-r+1)(\delta-r^2-r+1)-\frac{1}{2}(r^2-3)(r-1)\ (r\ \text{odd}\neq 3)$$

$$\dim_{\text{c.g.}}(r,d,\delta)=(d-r+1)\,(\delta-r^2-r+1)-\frac{1}{2}r^2(r-1).$$

Tables of dimensions for low degrees $r=\deg(f)$

degree r	*variable trivial*	*variable standard*	*variable odd*	*variable general*
1	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$
1	(2, 2)	(2, 4)	(2, 4)	(2, 5)
2	(3, 4)	(3, 14)	(3, 14)	(3, 15)
3	(4, 7)	(4, 28)	(4, 28)	(4, 34)
4	(5, 11)	(5, 64)	(5, 64)	(5, 65)
5	(6, 16)	(6, 100)	(6, 104)	(6, 111)
6	(7, 22)	(7, 174)	(7, 174)	(7, 175)
7	(8, 29)	(8, 244)	(8, 250)	(8, 260)
8	(9, 37)	(9, 368)	(9, 368)	(9, 369)
9	(10, 46)	(10, 484)	(10, 492)	(10, 505)
10	(11, 56)	(11, 670)	(11, 670)	(11, 671)
…	…	…	…	…

degree r	*variable trivial*	*variable standard*	*variable odd*	*variable general*
1	$(\overline{d},\underline{\delta})$	$(\overline{d},\underline{\delta})$	$(\overline{d},\underline{\delta})$	$(\overline{d},\underline{\delta})$
1	(3, 1)	(2, 4)	(2, 4)	(3, 4)
2	(5, 2)	(9, 8)	(9, 8)	(9, 9)
3	(8, 3)	(16, 16)	(16, 16)	(22, 16)
4	(12, 4)	(45, 24)	(45, 24)	(45, 25)
5	(17, 5)	(70, 36)	(74, 36)	(81, 36)
6	(23, 6)	(133, 48)	(133, 48)	(133, 49)
7	(30, 7)	(188, 64)	(194, 64)	(204, 64)
8	(38, 8)	(297, 80)	(297, 80)	(297, 81)
9	(47, 9)	(394, 100)	(402, 100)	(415, 100)
10	(57, 10)	(561, 120)	(561, 120)	(561, 121)
…	…	…	…	…

degree r	*covariant trivial*	*covariant standard*	*covariant odd*	*covariant general*
1	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$
1	(1, 1)	(1, 2)	(1, 2)	(1, 2)
2	(2, 2)	(2, 8)	(2, 8)	(2, 8)
3	(3, 4)	(3, 16)	(3, 21)	(3, 21)
4	(4, 7)	(4, 44)	(4, 44)	(4, 44)
5	(5, 11)	(5, 70)	(5, 80)	(5, 80)
6	(6, 16)	(6, 132)	(6, 132)	(6, 132)
7	(7, 22)	(7, 188)	(7, 203)	(7, 203)
8	(8, 29)	(8, 296)	(8, 296)	(8, 296)
9	(9, 37)	(9, 394)	(9, 414)	(9, 414)
10	(10, 46)	(10, 560)	(10, 560)	(10, 560)
...	...	...	...	...

degree r	*covariant trivial*	*covariant standard*	*covariant odd*	*covariant general*
1	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$
1	(1, 1)	(1, 2)	(1, 2)	(1, 2)
2	(2, 2)	(4, 6)	(4, 6)	(4, 6)
3	(4, 3)	(7, 12)	(7, 12)	(12, 12)
4	(7, 4)	(28, 20)	(28, 20)	(28, 20)
5	(11, 5)	(45, 30)	(49, 30)	(55, 30)
6	(16, 6)	(96, 42)	(96, 42)	(96, 42)
7	(22, 7)	(139, 56)	(145, 56)	(154, 56)
8	(29, 8)	(232, 72)	(232, 72)	(232, 72)
9	(37, 9)	(313, 90)	(321, 90)	(333, 90)
10	(46, 10)	(460, 110)	(460, 110)	(460, 110)
...	...	...	...	...

Differential polynomial P in the noncommuting variables (n, ν)
Our differential operators will be written as polynomials $P(n, \nu)$ of degree (d, δ) in the non-commuting variables (n, ν), which are capable of two realisations:

$$(n, \nu) \longrightarrow (n, -\partial_n) \quad or \quad (\partial_\nu, \nu).$$

Both realisation are of course compatible with $[n, \nu] = 1$ and the ODE interpretation goes like this:

$$P(n, -\partial_n)k(n) = 0 \iff P(\partial_\nu, \nu) \hat{k}(\nu) = P(\partial_\nu, \nu)\partial_\nu h(\nu) = 0. \quad (6.34)$$

Compressing the covariant ODEs
To get more manageable expressions, we can take advantage of the covariance relation to express everything in terms of shift-invariant data. This involves three steps:

(i) apply the above the ODE-finding algorithm of Section 6.1 to a centered polynomial $f(x) = \sum_{i=0}^{r-2} f_i\, x^i + f_r\, x^r$;
(ii) replace the coefficients $\{f_0, f_1, \ldots, f_{r-2}, f_r\}$ by the shift-invariants $\{\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2}, \mathbf{f}_r\}$ defined in Section 6.2 *infra*;
(iii) replace the β-coefficients by the "centered" $\boldsymbol{\beta}$-coefficients defined *infra*.

Basic polynomials $f(x)$ and $p(\nu)$

$$f(x) = f_0 + f_1 x + \ldots f_r x^r = (x - x_1)\ldots(x - x_r)\, f_r \tag{6.35}$$

$$p(\nu) = p_0 + p_1 \nu + \ldots p_r \nu^r = (\nu - \nu_1)\ldots(\nu - \nu_r)\, p_r \tag{6.36}$$

$$\text{with} \quad \nu_i = f^*(x_i) = \int_0^{x_i} f(x)dx = \sum_{0\le s\le r} f_s \frac{x_i^{s+1}}{s+1}. \tag{6.37}$$

The polynomials $p(\nu)$ are usually normalised by the condition $p_r = 1$ and their zeros ν_i are exactly the images under f^* of the zeros x_i of the input polynomial f. Since these ν_i correspond to the singular points of the inner generators in the ν-plane, we should expect the polynomials $p(\nu)$ to be a crucial ingredient of our ODEs. This is indeed the case – they will appear, predictably enough, as coefficients of the leading derivative.[58]

Basic symmetric functions $x_s^*, x_s^{**}, \nu_s^*, \nu_s^{**}$

$$x_1^* := \sum_{1\le i\le r} x_i, \qquad x_2^* := \sum_{1\le i<j\le r} x_i\, x_j \quad , \ldots, \quad x_r^* := x_1 \ldots x_r \tag{6.38}$$

$$\nu_1^* := \sum_{1\le i\le r} \nu_i, \qquad \nu_2^* := \sum_{1\le i<j\le r} \nu_i\, \nu_j \quad , \ldots, \quad \nu_r^* := \nu_1 \ldots \nu_r \tag{6.39}$$

$$x_s^{**} := \sum_{1\le i\le r} x_i^s \qquad (\forall s \in \mathbb{N}) \tag{6.40}$$

$$\nu_s^{**} := \sum_{1\le i\le r} \nu_i^s \qquad (\forall s \in \mathbb{N}). \tag{6.41}$$

[58] With the notations 6.34, this means that $p(\nu)$ is going to accompany the highest power of n in the non-commutative polynomial $P(n, \nu)$.

The change from the x-data to the ν-data goes like this:

$$\{f_s\} \longrightarrow \{x_s^*\} \xrightarrow{\text{i}} \{x_s^{**}\} \xrightarrow{\text{ii}} \{\nu_s^{**}\} \xrightarrow{\text{iii}} \{\nu_s^*\} \longrightarrow \{p_s\}$$

$$\text{(i)} \qquad \sum_{1\le s\le\infty} \frac{1}{s}\frac{x_s^{**}}{x^s} \equiv -\log\left(1+\sum_{1\le s\le r}(-1)^r\frac{x_s^*}{x^s}\right)$$

$$\text{(ii)} \qquad \nu_s^{**} \equiv \sum_{s\le t\le(r+1)s} f_{s,t}^*\, x_t^{**} \quad \text{with} \quad \sum_{s\le t\le(r+1)s} f_{s,t}^*\, x^t := (f^*(x))^s$$

$$\text{(iii)} \qquad 1+\sum_{1\le s\le r}(-1)^r\frac{\nu_s^*}{\nu^s} \equiv \exp\left(-\sum_{1\le s\le\infty}\frac{1}{s}\frac{\nu_s^{**}}{\nu^s}\right).$$

Centered polynomials. Invariants

$$x_0 := \frac{1}{r}(x_1+\cdots+x_r) = -\frac{1}{r}\frac{f_{r-1}}{f_r}$$

$$\nu_0 := f^*(x_0) = \int_0^{x_0} f(x)\,dx = \sum_{0\le s\le r} f_s\frac{x_0^{s+1}}{s+1}$$

$$\underline{\nu}_0 := \frac{1}{r}(\nu_1+\cdots+\nu_r) = -\frac{1}{r}\frac{p_{r-1}}{p_r} \qquad (\nu_0 \ne \underline{\nu}_0 \text{ in general})$$

$$\mathbf{f}(x) := f(x+x_0) = \sum_{0\le s\le r}\mathbf{f}_s x^s \quad (\mathbf{f}_{r-1}=0)$$

$$\mathbf{p}(\nu) := p(\nu+\nu_0) = \sum_{0\le s\le r}\mathbf{p}_s\nu^s \qquad \mathbf{P}(\nu) := P(\nu+\nu_0)$$

$$\underline{\mathbf{p}}(\nu) := p(\nu+\underline{\nu}_0) = \sum_{0\le s\le r}\underline{\mathbf{p}}_s\,\nu^s \quad (\underline{\mathbf{p}}_{r-1}=0) \qquad \underline{\mathbf{P}}(\nu) := P(\nu+\underline{\nu}_0)$$

Centered β-coefficients

$$\beta(\tau) = \tau^{-1}+\sum_{0\le k}\beta_k\,\tau^k = \tau^{-1}\left(1+\sum_{1\le k}\frac{b_k}{k!}\tau^k\right) \tag{6.42}$$

$$1+\sum_{2\le k}\frac{\mathbf{b}_k}{k!}\tau^k = \left(1+\sum_{1\le k}\frac{b_k}{k!}\tau^k\right)\left(1+\sum_{1\le k}\frac{(-b_1)^k}{k!}\tau^k\right) \tag{6.43}$$

$$\begin{aligned}
\mathbf{b}_1 &= 0 = 0\\
\mathbf{b}_2 &= b_2 - b_1^2 = 2\,\beta_1 - \beta_0^2\\
\mathbf{b}_3 &= b_3 - 3\,b_1b_2 + 2\,b_1^3 = 6\,\beta_2 - 6\,\beta_0\beta_1 + 2\,\beta_0^3\\
\mathbf{b}_4 &= b_4 - 4\,b_1b_3 + 6\,b_1^2b_2 - 3\,b_1^4 = 24\,\beta_3 - 24\,\beta_0\beta_2 + 12\,\beta_0^2\beta_1 - 3\,\beta_0^4\\
\mathbf{b}_5 &= b_5 - 5\,b_1b_4 + 10\,b_1^2b_3 - 10\,b_1^3b_2 + 4\,b_1^5\\
&= 120\,\beta_4 - 120\,\beta_0\beta_3 + 60\,\beta_0^2\beta_2 - 20\,\beta_0^3\beta_1 + 4\,\beta_0^5.
\end{aligned}$$

Invariance and homogeneousness under $f(\bullet) \mapsto \lambda f(\gamma \bullet + \epsilon)$
Invariance under $f(\bullet) \mapsto f(\bullet + \epsilon)$.

$$
\begin{array}{lll}
(x, n, \nu) \overset{\partial_\epsilon}{\mapsto} & (1, 0, -f_0) & \\
\partial_\epsilon x_i = -1 & (1 \le i \le r) & \partial_\epsilon x_0 = -1 \\
\partial_\epsilon \nu_i = -f_0 & (1 \le i \le r) & \partial_\epsilon \nu_0 = \partial_\epsilon \underline{\nu}_0 = -f_0 \\
\partial_\epsilon f_s = (1+s) f_{1+s} & (0 \le s < r)\ \partial_\epsilon f_r = 0 & \partial_\epsilon \mathbf{f}_s = 0\ \ (0 \le s \le r) \\
\partial_\epsilon p_s = (1+s) p_{1+s} f_0 & (0 \le s < r)\ \partial_\epsilon p_r = 0 & \partial_\epsilon \mathbf{p}_s = 0\ \ (0 \le s \le r).
\end{array}
$$

Homogeneousness under $f(\bullet) \mapsto f(\gamma \bullet)$.

$$
\begin{aligned}
(x, n, \nu) &\mapsto (\gamma^{-1} x, \gamma n, \gamma^{-1} \nu) \\
(f_s, \mathbf{f}_s) &\mapsto (\gamma^s f_s, \gamma^s \mathbf{f}_s) \\
(p_s, \mathbf{p}_s) &\mapsto (\gamma^{s-r} p_s, \gamma^{s-r} \mathbf{p}_s).
\end{aligned}
$$

Homogeneousness under $f(\bullet) \mapsto \lambda f(\bullet)$.

$$
\begin{aligned}
(x, n, \nu) &\mapsto (x, \lambda^{-1} n, \lambda \nu) \\
(f_s, \mathbf{f}_s) &\mapsto (\lambda f_s, \lambda \mathbf{f}_s) \\
(p_s, \mathbf{p}_s) &\mapsto (\lambda^{r-s} p_s, \lambda^{r-s} \mathbf{p}_s) \\
\beta_{s-1} &\mapsto \lambda^{-s} \beta_{s-1}.
\end{aligned}
$$

6.3 Explicit ODEs for low-degree polynomial inputs f

To avoid glutting this section, we shall restrict ourselves to the standard choice for β and mention only the covariant ODEs.[59] Concretely, for all values of the f-dregree r up to 4 we shall write down a complete set of *minimal* polynomials $P_{(d_i,\delta_i)}(n, \nu)$, of degrees (d_i, δ_i) in (n, ν), that generate all the other convariant polynomials by non-commutative pre-multiplication by covariant polynomials in (n, ν).[60] For each r, the sequence

$$(\underline{d}, \overline{\delta})^1, \ldots, (d_i, \delta_i)^{m_i}, \ldots, (\overline{d}, \underline{\delta})^1$$

indicates the degrees (d_i, δ_i) of all minimal spaces with their dimensions m_i, *i.e.* the number of polynomials in them. For the extreme cases, right and left, that dimension is always 1.

[59] But we keep extensive tables for all 8 cases $(v, c) \times (t, s, o, g)$ at the disposal of the interested reader.

[60] In fact, all *variable* ODEs can also be expressed as suitable combinations of the minimal *covariant* ODEs.

Input f of degree 1
Invariant coefficients: $\mathbf{f}_1 := f_1$.
Covariant shift: $\nu_0 := -\frac{1}{2}\frac{f_0^2}{f_1}$.
First leading polynomial (shifted): $\mathbf{p}(\nu) = p(\nu+\nu_0) = \nu$.
Second leading polynomial: $\mathbf{q}(n) = n^2$.
Covariant differential equations:(1, 2)

$$\mathbf{P}_{(1,2)}(n,\nu) = P_{(1,2)}(n,\nu+\nu_0) = n^2\,\nu + \frac{1}{2}\,n - \frac{1}{24}\,\mathbf{f}_1.$$

Variable differential equations:(2, 4).
Input f of degree 2
Invariant coefficients:

$$\mathbf{f}_0 = f_0 - \frac{1}{4}\frac{f_1^2}{f_2} = -\frac{1}{4}\,(x_1-x_2)^2\,f_2, \quad \mathbf{f}_2 = f_2.$$

Covariant shift:

$$\nu_0 = -\frac{1}{2}\frac{f_0 f_1}{f_2} + \frac{1}{12}\frac{f_1^3}{f_2^2} = -\frac{1}{12}\,(x_1+x_2)(x_1^2-4x_1x_2+x_2^2)\,f_2.$$

Leading scalar factor:

$$\mathbf{f}_0 = -\frac{1}{4}\,(x_1-x_2)^2\,f_2.$$

First leading polynomial (shifted)

$$\mathbf{p}(\nu) = p(\nu+\nu_0) = \frac{4}{9}\frac{\mathbf{f}_0^3}{\mathbf{f}_2} + \nu^2.$$

Second leading polynomial:

$$\mathbf{q}(n) = \frac{1}{6}\frac{\mathbf{f}_2}{\mathbf{f}_0}n^6 + n^8.$$

Covariant differential equations: (2, 8), $(3, 7)^2$, (4, 6)

$$\begin{aligned}\mathbf{P}_{(2,8)}(n,\nu) &= P_{(2,8)}(n,\nu+\nu_0) = n^8\,\mathbf{f}_0\,\mathbf{p}(\nu) + n^7\,\mathbf{f}_0\,\nu\\ &+ n^6\left(\frac{1}{6}\mathbf{f}_2\,\nu^2 + \frac{5}{27}\mathbf{f}_0^3 + \frac{8}{9}\mathbf{f}_0\right)\\ &- n^5\left(\frac{1}{6}\mathbf{f}_2\,\nu\right) + n^4\left(\frac{1}{54}\mathbf{f}_0^2\,\mathbf{f}_2 - \frac{2}{27}\mathbf{f}_2\right) - n^2\left(\frac{1}{972}\mathbf{f}_0\,\mathbf{f}_2^2\right) - \frac{1}{583}\,\mathbf{f}_2^3\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{(3,7)}(n,\nu) &= P_{(3,7)}(n,\nu+\nu_0) = n^7\,\mathbf{f}_0\,\mathbf{p}(\nu)\\
&+ n^6\left(-\frac{1}{32}\mathbf{f}_2\,\nu^3 - \frac{1}{72}\mathbf{f}_0^3\,\nu + \mathbf{f}_0\,\nu\right)\\
&+ n^5\left(\frac{7}{32}\mathbf{f}_2\,\nu^2 + \frac{8}{9}\mathbf{f}_0 + \frac{2}{9}\mathbf{f}_0^3\right) + n^4\left(-\frac{1}{288}\mathbf{f}_2\,\mathbf{f}_0^2\nu - \frac{41}{288}\mathbf{f}_2\,\nu\right)\\
&+ n^3\left(\frac{7}{432}\mathbf{f}_0^2\,\mathbf{f}_2 - \frac{1}{18}\mathbf{f}_2\right) - n\left(\frac{11}{7776}\mathbf{f}_0\,\mathbf{f}_2^2\right) + \frac{1}{31104}\,\mathbf{f}_2^3\,\nu
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}^\dagger_{(3,7)}(n,\nu) &= P^\dagger_{(3,7)}(n,\nu+\nu_0) = n^7\,\mathbf{f}_0\,\nu\,\mathbf{p}(\nu) + n^6\left(-\frac{5}{3}\mathbf{f}_0\,\nu^2 - \frac{32}{27}\frac{\mathbf{f}_0^4}{\mathbf{f}_2}\right)\\
&+ n^5\left(-\frac{7}{9}\mathbf{f}_0\,\nu + \frac{1}{9}\mathbf{f}_0^3\,\nu\right) + n^4\left(\frac{2}{27}\mathbf{f}_0^3 - \frac{16}{27}\mathbf{f}_0\right)\\
&+ n^2\left(\frac{1}{81}\mathbf{f}_0^2\,\mathbf{f}_2\right) - n\left(\frac{1}{972}\mathbf{f}_0\,\mathbf{f}_2^2\nu\right) - \frac{4}{729}\,\mathbf{f}_0\,\mathbf{f}_2^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{(4,6)}(n,\nu) &= P_{(4,6)}(n,\nu+\nu_0) = n^6\,\mathbf{f}_0\left(\nu^2 + \frac{416}{3}\frac{\mathbf{f}_0}{\mathbf{f}_2}\right)\mathbf{p}(\nu)\\
&+ n^5\left(-13\mathbf{f}_0\,\nu^3 + \frac{416}{3}\frac{\mathbf{f}_0^2}{\mathbf{f}_2}\,\nu - \frac{56}{9}\frac{\mathbf{f}_0^4}{\mathbf{f}_2}\,\nu\right)\\
&+ n^4\left(\frac{356}{9}\mathbf{f}_0\,\nu^2 + \frac{1}{9}\mathbf{f}_0^3\,\nu^2 + \frac{3328}{27}\frac{\mathbf{f}_0^2}{\mathbf{f}_2} + \frac{1024}{27}\frac{\mathbf{f}_0^4}{\mathbf{f}_2}\right)\\
&+ n^3\left(-\frac{148}{9}\mathbf{f}_0\,\nu - \frac{26}{27}\mathbf{f}_0^3\,\nu\right) + n^2\left(\frac{158}{81}\mathbf{f}_0^3 - \frac{16}{3}\mathbf{f}_0\right)\\
&+ n\left(\frac{1}{81}\mathbf{f}_0^2\,\mathbf{f}_2\nu\right) - \frac{1}{972}\,\mathbf{f}_0\,\mathbf{f}_2^2\,\nu^2 - \frac{161}{729}\mathbf{f}_0^2\,\mathbf{f}_2.
\end{aligned}$$

Variable differential equations: $(3, 14)$, $(4, 11)^2$, $(5, 10)^3$, $(6, 9)^2$, $(9, 8)$

Input f of degree 3

Invariant coefficients:

$$\begin{aligned}
\mathbf{f}_0 &= f_0 - \frac{1}{3}\frac{f_1\,f_2}{f_3} + \frac{2}{27}\frac{f_2^3}{f_3^2}\\
&= \frac{1}{27}(x_1 + x_2 - 2\,x_3)(x_2 + x_3 - 2\,x_1)(x_3 + x_1 - 2\,x_2)\,f_3\\
\mathbf{f}_1 &= f_1 - \frac{1}{3}\frac{f_2^2}{f_3} = -\frac{1}{3}(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)\,f_3\\
\mathbf{f}_3 &= f_3.
\end{aligned}$$

Covariant shift:

$$\nu_0 = -\frac{1}{3}\frac{f_0 f_2}{f_3} \Big/ + \frac{1}{18}\frac{f_1 f_2^2}{f_3^2} - \frac{1}{108}\frac{f_2^4}{f_3^3} = -\frac{1}{108}(x_1+x_2+x_3)(x_1^3+x_2^3+x_3^3 \\ +24x_1x_2x_3 - 3x_1x_2^2 - 3x_1^2x_2 - 3x_3x_1^2 - 3x_3^2x_1 - 3x_3x_2^2 - 3_3^2x_2)\, f_3.$$

Leading scalar factor:

$$\mathbf{a} := 4\,\mathbf{f}_1^3 + 27\,\mathbf{f}_0^2\,\mathbf{f}_3 = -(x_1 - x_2)^2\,(x_2 - x_3)^2\,(x_3 - x_2)^2\, f_3^3.$$

First leading polynomial (shifted)

$$\mathbf{p}(\nu) = p(\nu + \nu_0) = \frac{1}{32}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3^2} + \frac{27}{64}\frac{\mathbf{f}_0^4}{\mathbf{f}_3} + \left(\frac{9}{8}\frac{\mathbf{f}_0^2\mathbf{f}_1}{\mathbf{f}_3} + \frac{1}{16}\frac{\mathbf{f}_1^4}{\mathbf{f}_3^2}\right)\nu + \frac{1}{2}\frac{\mathbf{f}_1^2}{\mathbf{f}_3}\nu^2 + \nu^3.$$

Second leading polynomial:

$$\mathbf{q}(n) = \frac{9}{4}\frac{\mathbf{f}_1\,\mathbf{f}_3^2}{\mathbf{a}}\,n^{12} + \frac{81}{4}\frac{\mathbf{f}_3^2}{\mathbf{a}}\,n^{13} + 6\,\frac{\mathbf{f}_1^2\,\mathbf{f}_3}{\mathbf{a}}\,n^{14} + 3\,\frac{\mathbf{f}_1\,\mathbf{f}_3}{\mathbf{a}}\,n^{15} + n^{16}.$$

Covariant differential equations: $(3, 16)$, $(4, 14)^2$, $(5, 13)^2$, $(7, 12)$

$$\begin{aligned}
\mathbf{P}_{(3,16)}(n,\nu) &= P_{(3,16)}(n,\nu+\nu_0) = \mathbf{a}\,\mathbf{p}(\nu) + O(n^{15})\,O(\nu^3)\\
\mathbf{P}_{(4,14)}(n,\nu) &= P_{(4,14)}(n,\nu+\nu_0) = \mathbf{f}_1\,\mathbf{a}\,\mathbf{b}\,n^{14}\,p(\nu) + O(n^{13})\,O(\nu^4)\\
\mathbf{P}^{\dagger}_{(4,14)}(n,\nu) &= P^{\dagger}_{(4,14)}(n,\nu+\nu_0) = \mathbf{a}\,\mathbf{b}\,n^{14}\nu\,p(\nu) + O(n^{13})\,O(\nu^4)
\end{aligned}$$

with the following invariant coefficient **b**:

$$\begin{aligned}
\mathbf{b} := {} & 2097152\,\mathbf{f}_1^{12} - 766779696\,\mathbf{f}_1^3\,\mathbf{f}_3^3 - 520497152\,\mathbf{f}_1^9\,\mathbf{f}_3\\
& - 36074005128\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^3 + 1428879744\,\mathbf{f}_1^6\,\mathbf{f}_3^2\\
& - 1314579456\,\mathbf{f}_0^6\mathbf{f}_1^3\,\mathbf{f}_3^3 + 1099865088\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^2\\
& + 205963264\,\mathbf{f}_1^9\,\mathbf{f}_0^2\,\mathbf{f}_3 - 8872609536\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^2\\
& + 73222472421\,\mathbf{f}_0^4\,\mathbf{f}_3^4 + 20602694736\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^3\\
& + 5971968\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^2 + 884736\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3 - 5165606520\,\mathbf{f}_0^2\,\mathbf{f}_3^4
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{(5,13)}(n,\nu) &= P_{(5,13)}(n,\nu+\nu_0)\\
&= n^{13}\,(\mathbf{f}_1^2\,\mathbf{c}_1 - 180\,\mathbf{f}_3\,\mathbf{c}_2\,\nu)\,\mathbf{p}(\nu) + O(n^{12})\,O(\nu^5)\\
\mathbf{P}^{\dagger}_{(5,13)}(n,\nu) &= P^{\dagger}_{(5,13)}(n,\nu+\nu_0)\\
&= n^{13}\,(\mathbf{c}_3\,\nu + 180\,\mathbf{f}_1\,\mathbf{f}_3\,\mathbf{c}_1\,\nu^2)\,\mathbf{p}(\nu) + O(n^{12})\,O(\nu^5)
\end{aligned}$$

with the following invariant coefficients $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$:

$$
\begin{aligned}
\mathbf{c}_1 := {} & 917290620205793280\,\mathbf{f}_0^2\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 + 787176090501 12\,\mathbf{f}_0^6\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& + 4163751641088\,\mathbf{f}_0^4\,\mathbf{f}_1^{15}\,\mathbf{f}_3 + 50281437903388672\,\mathbf{f}_1^{15}\,\mathbf{f}_3 \\
& + 1581069280739328\,\mathbf{f}_0^2\,\mathbf{f}_1^{15}\,\mathbf{f}_3 + 17755411807125504\,\mathbf{f}_0^4\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& + 99407759207731200\,\mathbf{f}_0^6\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 564080018165226777 6\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
& + 344140580192256\,\mathbf{f}_0^8\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 1172666955060657043 2\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 3265897813810421 76\,\mathbf{f}_0^8\,\mathbf{f}_1^6\,\mathbf{f}_3^4 - 4984963478435306 88\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 16926659444736\,\mathbf{f}_0^{10}\,\mathbf{f}_1^6\,\mathbf{f}_3^4 - 85405328111733120\,\mathbf{f}_0^8\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& - 1691608028258304\,\mathbf{f}_0^{10}\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 1539050918501843226 0\,\mathbf{f}_0^6\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 98766738625551624\,\mathbf{f}_0^8\,\mathbf{f}_3^6 - 743253702832987862 4\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 10331678048256\,\mathbf{f}_1^{18} - 2731996121355095035 5\,\mathbf{f}_0^4\,\mathbf{f}_3^6 \\
& + 15007175850454416 00\,\mathbf{f}_0^2\,\mathbf{f}_3^6 + 2269603755161316 00\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& - 88258622384581632\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 - 8253051882421560660\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& - 1478991931831367424\,\mathbf{f}_0^2\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 450793967617700928\,\mathbf{f}_3^3\,\mathbf{f}_1^9 \\
& + 3821964710670454374\,\mathbf{f}_0^6\,\mathbf{f}_3^6 - 17792355610879876332\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& - 5761805211034236864\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
\mathbf{c}_2 := {} & 3460562666533471 68\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 561701191680\,\mathbf{f}_1^{18} \\
& - 1249214016002457 6\,\mathbf{f}_0^2\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 - 1568573227008\,\mathbf{f}_0^6\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& - 225501511680\,\mathbf{f}_0^4\,\mathbf{f}_1^{15}\,\mathbf{f}_3 - 812740325605376\,\mathbf{f}_1^{15}\,\mathbf{f}_3 \\
& - 83614219370496\,\mathbf{f}_0^2\,\mathbf{f}_1^{15}\,\mathbf{f}_3 - 1675724924436480\,\mathbf{f}_0^4\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& - 7670187447717888\,\mathbf{f}_0^6\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 50653665706906368\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
& - 313456656384\,\mathbf{f}_0^8\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 23099325980303808\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 24015789981646272\,\mathbf{f}_1^6\,\mathbf{f}_3^4 + 28152325951488\,\mathbf{f}_0^8\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& - 64268410079232\,\mathbf{f}_0^8\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 24631997881011588\,\mathbf{f}_0^6\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 125078862747499267 5\,\mathbf{f}_0^4\,\mathbf{f}_3^6 + 308789626552560\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 1527911696305152\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 + 602232106994037 00\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 198638494195392 00\,\mathbf{f}_0^2\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 18320806414937664\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
& + 3706040377703682\,\mathbf{f}_0^6\,\mathbf{f}_3^6 + 945719068633781580\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 263746805911956288\,\mathbf{f}_0^2 - g f_1^6\,\mathbf{f}_3^4
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_3 := {} & 188945409245184\,\mathbf{f}_1^{21} - 4265434334643431940864\,\mathbf{f}_0^{8}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& + 10475616970801152\,\mathbf{f}_0^{8}\,\mathbf{f}_1^{12}\,\mathbf{f}_3^{3} \\
& - 64097798636847956405 76\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{12}\,\mathbf{f}_3^{3} \\
& + 16435859799336647523 84\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{12}\,\mathbf{f}_3^{3} \\
& + 31374503650787328\,\mathbf{f}_0^{10}\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& + 397764497706025665 8880\,\mathbf{f}_0^{6}\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& + 1290160568497752489024\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& + 1205510242413389496768\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& - 4840569984394946992 0500\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& + 129087554262282601920\,\mathbf{f}_1^{12}\,\mathbf{f}_3^{3} \\
& - 280615966839399140352\,\mathbf{f}_1^{15}\,\mathbf{f}_3^{2} \\
& + 9973443990092156928\,\mathbf{f}_0^{6}\,\mathbf{f}_1^{12}\,\mathbf{f}_3^{3} \\
& - 1492256344300529883948\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& + 914039610015744\,\mathbf{f}_0^{12}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& - 12291903327944756821 4604\,\mathbf{f}_0^{6}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& + 2957452975344634 6752\,\mathbf{f}_0^{8}\,\mathbf{f}_1^{9}\,\mathbf{f}_3^{4} \\
& + 132978955491577 03680\,\mathbf{f}_0^{10}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& + 3031199240275539529 6032\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& - 2476350254753930756 4426\,\mathbf{f}_0^{6}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& + 169418141509536645120\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{15}\,\mathbf{f}_3^{2} \\
& - 908625541799649020400\,\mathbf{f}_1^{6}\,\mathbf{f}_3^{5} \\
& + 56035330510879671 84\,\mathbf{f}_0^{10}\,\mathbf{f}_3^{7} - 678375642665334 09024\,\mathbf{f}_0^{10}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& + 3613883506978117107600\,\mathbf{f}_0^{8}\,\mathbf{f}_3^{7} \\
& - 4102586698117975974 0000\,\mathbf{f}_0^{4}\,\mathbf{f}_3^{7} \\
& + 5812866888802370809 92900\,\mathbf{f}_0^{6}\,\mathbf{f}_3^{7} \\
& - 1221065965233634266 7200\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& + 2319375004590101113 67085\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& - 1297732672262124504 5184\,\mathbf{f}_0^{8}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& + 10494104263821066 24\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{15}\,\mathbf{f}_3^{2} - 10060148457735782 4\,\mathbf{f}_0^{12}\,\mathbf{f}_1^{3}\,\mathbf{f}_3^{6} \\
& + 27598162056708096\,\mathbf{f}_0^{2}\,\mathbf{f}_1^{18}\,\mathbf{f}_3 + 76397618921472\,\mathbf{f}_0^{4}\,\mathbf{f}_1^{18}\,\mathbf{f}_3 \\
& + 400879420000010240 0\,\mathbf{f}_1^{18}\,\mathbf{f}_3 + 1381995569479680\,\mathbf{f}_0^{6}\,\mathbf{f}_1^{15}\,\mathbf{f}_3^{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}_{(7,12)}(n,\nu) &= P_{(7,12)}(n,\nu+\nu_0) \\
&= n^{12}\,p(\nu)\,(\mathbf{f}_1^{4}\,\nu^4 + \mathbf{d}_3\,\nu^3 + \mathbf{f}_1^{2}\,\mathbf{d}_2\,\nu^2 + \mathbf{f}_1\,\mathbf{d}_1\,\nu + \mathbf{d}_0) + O(n^{11})\,O(\nu^7)
\end{aligned}
$$

with the following invariant coefficients $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$:

$$\mathbf{d}_0 := +\frac{29859111}{128}\mathbf{f}_0^2 + \frac{3664683}{1600}\mathbf{f}_0^4 + \frac{29889}{4000}\mathbf{f}_0^6 + \frac{81}{10000}\mathbf{f}_0^8 + \frac{22240737}{640}\frac{\mathbf{f}_1^3}{\mathbf{f}_3}$$
$$+\frac{336626989}{3200}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{3493333}{24000}\frac{\mathbf{f}_0^4\mathbf{f}_1^3}{\mathbf{f}_f 3} + \frac{159}{5000}\frac{\mathbf{f}_1^3\mathbf{f}_0^6}{\mathbf{f}_3} + \frac{1969}{60000}\frac{\mathbf{f}_0^4\mathbf{f}_1^6}{\mathbf{f}_3^2}$$
$$+\frac{242977752829}{15552000}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{40541647}{1296000}\frac{\mathbf{f}_0^2\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{15317}{4860000}\frac{\mathbf{f}_0^2\mathbf{f}_1^9}{\mathbf{f}_3^3}$$
$$+\frac{203363491}{69984000}\frac{\mathbf{f}_1^9}{\mathbf{f}_3^3} + \frac{83521}{1049760000}\frac{\mathbf{f}_1^{12}}{\mathbf{f}_3^4}$$

$$\mathbf{d}_1 := -\frac{368631}{160} + \frac{3305043}{800}\mathbf{f}_0^2 + \frac{93339}{2000}\mathbf{f}_0^4$$
$$+\frac{27}{250}\mathbf{f}_0^6 - \frac{642277459}{1296000}\frac{\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{123}{500}\frac{\mathbf{f}_0^4\mathbf{f}_1^3}{\mathbf{f}_3}$$
$$+\frac{10657943}{18000}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{697}{9000}\frac{\mathbf{f}_0^2\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{101072021}{1944000}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{4913}{1458000}\frac{\mathbf{f}_1^9}{\mathbf{f}_3^3}$$

$$\mathbf{d}_2 := +\frac{361809}{800} + \frac{10467}{200}\mathbf{f}_0^2 + \frac{27}{50}\mathbf{f}_0^4 + \frac{479929}{2160}\frac{\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{29}{50}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{289}{5400}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2}$$

$$\mathbf{d}_3 := -\frac{6561}{40}\mathbf{f}_3 - \frac{1347}{20}\mathbf{f}_1^3 + \frac{6}{5}\mathbf{f}_0^2\mathbf{f}_1^3 + \frac{17}{45}\frac{\mathbf{f}_1^6}{\mathbf{f}_3}.$$

Variable differential equations:

$$(4, 28), (5, 22)^2, (6, 20)^3, (7, 19)^4, (8, 18)^3, (10, 17)^2, (16, 16).$$

Input f of degree 4
Invariant coefficients:

$$\mathbf{f}_0 = f0 - \frac{1}{4}\frac{f_1 f_3}{f4} + \frac{1}{16}\frac{f_2 f_3^2}{f_4^2} - \frac{3}{256}\frac{f_3^4}{f_4^3} = \frac{1}{256}\prod_{i=1}^{i=4}(4x_i - x_1 - x_2 - x_3 - x_4)$$

$$\mathbf{f}_1 = f_1 - \frac{1}{2}\frac{f_2 f_3}{f_4} + \frac{1}{8}\frac{f_3^3}{f_4^2}$$
$$= -\frac{1}{8}(x_1+x_2-x_3-x_4)(x_2+x_3-x_1-x_4)(x_2+x_4-x_1-x_3)\, f_4$$

$$\mathbf{f}_2 = f_2 - \frac{3}{8}\frac{f_3^2}{f_4}$$
$$= -\frac{1}{8}\left(4(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2\right) f_4$$

$$\mathbf{f}_4 = f_4.$$

Covariant shift:

$$\nu_0 = -\frac{1}{4}\frac{f_0 f_3}{f_4} + \frac{1}{32}\frac{f_1 f_3^2}{f_4^2} - \frac{1}{192}\frac{f_2 f_3^3}{f_4^3} + \frac{1}{1280}\frac{f_3^5}{f_4^4}.$$

Leading scalar factors:

$$\begin{aligned}\mathbf{a} &= 256\,\mathbf{f}_0^3\,\mathbf{f}_4^2 - 128\,\mathbf{f}_0^2\,\mathbf{f}_2^2\,\mathbf{f}_4 + 16\,\mathbf{f}_0\,\mathbf{f}_2^4 + 144\,\mathbf{f}_0\,\mathbf{f}_2\,\mathbf{f}_1^2\,\mathbf{f}_4 - 4\,\mathbf{f}_1^2\,\mathbf{f}_2^3 \\ &\quad - 27\,\mathbf{f}_1^4\,\mathbf{f}_4 = \prod_{1\le i<j\le 4} (x_i - x_j)^2\, f_4^5\end{aligned}$$

$$\begin{aligned}\mathbf{b} &= -1280\,\mathbf{f}_2^6 + 32256\,\mathbf{f}_0\,\mathbf{f}_2^4\,\mathbf{f}_4 - 269568\,\mathbf{f}_0^2\,\mathbf{f}_2^2\,\mathbf{f}_4^2 + 746496\,\mathbf{f}_0^3\,\mathbf{f}_4^3 \\ &\quad + 69984\,\mathbf{f}_0\,\mathbf{f}_1^2\,\mathbf{f}_2\,\mathbf{f}_4^2 - 9504\,\mathbf{f}_1^2\,\mathbf{f}_2^3\,\mathbf{f}_4 + 19683\,\mathbf{f}_1^4\,\mathbf{f}_4^2 \\ &= \prod_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4}}^{(i,j)\neq(k,l)} \frac{1}{128}\Big(5\,(x_i+x_j-x_k-x_l)^2+(x_i-x_j)^2-5\,(x_k-x_l)^2\Big)\, f_4^6.\end{aligned}$$

First leading polynomial (shifted)

$$\begin{aligned}\mathbf{p}(\nu) &= \frac{12}{125}\frac{\mathbf{f}_0^3\mathbf{f}_1^2\mathbf{f}_2}{\mathbf{f}_4^2} - \frac{27}{2000}\frac{\mathbf{f}_0^2\mathbf{f}_1^4}{\mathbf{f}_4^2} + \frac{256}{625}\frac{\mathbf{f}_0^5}{\mathbf{f}_4} + \frac{16}{2025}\frac{\mathbf{f}_0^3\mathbf{f}_2^4}{\mathbf{f}_4^3} \\ &\quad - \frac{128}{1125}\frac{\mathbf{f}_0^4\mathbf{f}_2^2}{\mathbf{f}_4^2} - \frac{1}{675}\frac{\mathbf{f}_0^2\mathbf{f}_1^2\mathbf{f}_2^3}{\mathbf{f}_4^3} + \Bigg(\frac{32}{25}\frac{\mathbf{f}_0^3\mathbf{f}_1}{\mathbf{f}_4} - \frac{56}{225}\frac{\mathbf{f}_0^2\mathbf{f}_1\mathbf{f}_2^2}{\mathbf{f}_4^2} \\ &\quad + \frac{21}{100}\frac{\mathbf{f}_0\mathbf{f}_1^3\mathbf{f}_2}{\mathbf{f}_4^2} - \frac{27}{1000}\frac{\mathbf{f}_1^5}{\mathbf{f}_4^2} + \frac{4}{225}\frac{\mathbf{f}_0\mathbf{f}_1\mathbf{f}_2^4}{\mathbf{f}_4^3} - \frac{2}{675}\frac{\mathbf{f}_1^3\mathbf{f}_2^3}{\mathbf{f}_4^3}\Bigg)\,\nu \\ &\quad + \Bigg(\frac{16}{15}\frac{\mathbf{f}_0^2\mathbf{f}_2}{\mathbf{f}_4} + \frac{9}{10}\frac{\mathbf{f}_0\mathbf{f}_1^2}{\mathbf{f}_4} + \frac{11}{60}\frac{\mathbf{f}_1^2\mathbf{f}_2^2}{\mathbf{f}_4^2} - \frac{4}{15}\frac{\mathbf{f}_0\mathbf{f}_2^3}{\mathbf{f}_4^2} + \frac{4}{225}\frac{\mathbf{f}_2^5}{\mathbf{f}_4^3}\Bigg)\nu^2 + \frac{\mathbf{f}_1\mathbf{f}_2}{\mathbf{f}_4}\nu^3 + \nu^4.\end{aligned}$$

Second leading polynomial:

$$\begin{aligned}\mathbf{q}(n) &= -\frac{2^{14}\,7}{3^3\,5^6}\frac{\mathbf{f}_4^{11}}{\mathbf{ab}}\,n^{20} - \frac{2^{13}\,11}{3^2\,5^5}\frac{\mathbf{f}_2\,\mathbf{f}_4^{10}}{\mathbf{a}\,\mathbf{b}}\,n^{22} + \ldots \\ &\quad + 8\left(\frac{\mathbf{f}_1\,\mathbf{f}_2^2\,\mathbf{f}_4}{\mathbf{a}} + 12\,\frac{\mathbf{f}_0\,\mathbf{f}_1\,\mathbf{f}_4^2}{\mathbf{a}}\right) n^{43} + n^{44}.\end{aligned}$$

Covariant differential equations:

$$(4,44),\ (5,32)^2,\ (6,28)^3,\ (7,26)^4,\ (8,24),$$
$$(10,23)^4,\ (11,22)^3,\ (16,21)^2,\ (28,20)$$

$$\mathbf{P}_{(4,44)}(n,\nu) = \mathbf{P}_{(4,44)}(n,\nu+\nu_0) = \mathbf{a}\,\mathbf{b}\,n^{44}\,\mathbf{p}(\nu) + O(n^{43})\,O(\nu^4).$$

Variable differential equations:

$$(5,64),(6,44)^2,(7,37)^2,(8,34)^4,(9,32)^5,(10,30)^2,$$
$$(11,29)^2,(13,28)^5,(15,27)^4,(18,26)^2,(25,25)^2,(45,24).$$

6.4 The global resurgence picture for polynomial inputs f

The covariant ODEs enable us to describe the exact singular behaviour of $\hat{k}(\nu) = h(\nu)$ at infinity in the ν-plane, and by way of consequence all singularities *over* 0 in the ζ-plane. In the ν-plane, the singularities in question consist of linear combinations of rather elementary exponential factors multiplied by series in negative powers of ν. These are always divergent, resurgent, and resummable. The case of radial inputs f (*i.e.* $f(x) = f_r\, x^r$) is predictably much simpler and deserves special mention. We find:

$$\left(\sum_{r+1\le k} c_s(\omega)\nu^{-\frac{s}{r+1}}\right) \quad \exp\left(\omega\,\nu^{\frac{r}{r+1}}\right) \qquad \text{(for radial } f) \tag{6.44}$$

$$\left(\sum_{r+1\le k} c_s(\omega)\nu^{-\frac{s}{r+1}}\right)\exp\left(\omega\,\nu^{\frac{r}{r+1}}+\sum_{s=1}^{r-2}\omega_s\,\nu^{\frac{s}{r+1}}\right) \quad \text{(for general } f). \tag{6.45}$$

The “leading” frequencies ω featuring in the exponential factors depend only on the leading coefficient f_r of f. Via the variable θ thus defined:

$$\theta := \left(\frac{r+1}{r}\right)^r \frac{\mathbf{f}_r}{\omega^{r+1}} = \left(\frac{r+1}{r}\right)^r \frac{f_r}{\omega^{r+1}} \tag{6.46}$$

the leading frequencies ω correspond, for each degree r, to the roots of the following polynomials $\boldsymbol{\pi}_r(\theta)$ of degree r:

$$\begin{aligned}
\boldsymbol{\pi}_1(\theta) &= -12+\theta\\
\boldsymbol{\pi}_2(\theta) &= -432+\theta^2 = -2^4\,3^3+\theta^2\\
\boldsymbol{\pi}_3(\theta) &= (240+7\,\theta)\,(-30+\theta)^2 = (2^4\,3\times5+7\,\theta)\,(-2\times3\times5+\theta)^2\\
\boldsymbol{\pi}_4(\theta) &= (1749600000-1620000\,\theta^2+343\,\theta^4)\\
&= (2^8\,3^7\,5^5-2^5\,3^4\,5^4\,\theta^2+7^3\,\theta^4)\\
\boldsymbol{\pi}_5(\theta) &= (-1344+31\,\theta)\,(189+\theta)^2\,(42+\theta)^2\\
&= (-2^6\,3\times7+31\,\theta)\,(3^3\,7+\theta)^2\,(2\times3\times7+\theta)^2\\
\boldsymbol{\pi}_6(\theta) &= (-66395327975424+152320630896\,\theta^2\\
&\quad -116688600\,\theta^4+29791\,\theta^6)\\
&= (-2^{12}\,3^9\,7^7+2^4\,3^7\,7^6\,37\,\theta^2-2^3\,3^5\,5^2\,7^4\,\theta^4+31^3\,\theta^6)\\
\boldsymbol{\pi}_7(\theta) &= (3840+127\,\theta)\,(-30+\theta)^2\,(24300+1080\,\theta+37\,\theta^2)^2\\
&= (2^8\,3\times5+127\,\theta)(-2\times3\times5+\theta)^2(2^2\,3^5\,5^2+2^3\,3^3\,5\theta+37\theta^2)^2.
\end{aligned}$$

For a non-standard choice of β and with the "centered" coefficients $\mathbf{b}_i$ introduced at the end of Section 6.2, these polynomials $\boldsymbol{\pi}_r(\theta)$ become:

$$\boldsymbol{\pi}_1(\theta) = 1 + \mathbf{b}_2\,\theta$$

$$\boldsymbol{\pi}_2\,(\theta) = 1 + 2\,\mathbf{b}_3\theta + (\mathbf{b}_3^2 + 4\,\mathbf{b}_2^3)\,\theta^2$$

$$\begin{aligned}\boldsymbol{\pi}_3(\theta) = {} & 1 + 3\,(\mathbf{b}_4 - 6\,\mathbf{b}_2^2)\,\theta + 3\,(\mathbf{b}_4^2 + 18\,\mathbf{b}_2\mathbf{b}_3^2 - 12\,\mathbf{b}_2^2\mathbf{b}_4 + 27\,\mathbf{b}_2^4)\,\theta^2 \\ & + (\mathbf{b}_4^3 - 27\,\mathbf{b}_3^4 + 54\,\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4 - 18\,\mathbf{b}_2^2\mathbf{b}_4^2 - 54\,\mathbf{b}_2^3\mathbf{b}_3^2 + 81\,\mathbf{b}_2^4\mathbf{b}_4)\,\theta^3\end{aligned}$$

$$\begin{aligned}\boldsymbol{\pi}_4(\theta) = {} & 1 + 4\,(\mathbf{b}_5 - 30\,\mathbf{b}_2\mathbf{b}_3)\,\theta \\ & + 2\,(3\,\mathbf{b}_5^2 + 80\,\mathbf{b}_2\mathbf{b}_4^2 + 180\,\mathbf{b}_3^2\,\mathbf{b}_4 - 180\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5 \\ & + 1320\,\mathbf{b}_2^2\mathbf{b}_3^2 - 720\,\mathbf{b}_2^3\mathbf{b}_4 + 1728\,\mathbf{b}_2^5)\,\theta^2 + 4\,(\mathbf{b}_5^3 - 160\,\mathbf{b}_3\mathbf{b}_4^3 \\ & - 90\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5^2 + 180\,\mathbf{b}_3^2\mathbf{b}_4\mathbf{b}_5 + 80\,\mathbf{b}_2\mathbf{b}_4^2\mathbf{b}_5 \\ & + 1120\,\mathbf{b}_2^2\mathbf{b}_3\mathbf{b}_4^2 + 864\,\mathbf{b}_3^5 \\ & - 2520\,\mathbf{b}_2\mathbf{b}_3^3\mathbf{b}_4 + 1320\,\mathbf{b}_2^2\mathbf{b}_3^2\mathbf{b}_5 - 720\,\mathbf{b}_2^3\mathbf{b}_4\mathbf{b}_5 \\ & + 1728\,\mathbf{b}_2^5\mathbf{b}_5 + 1280\,\mathbf{b}_2^3\mathbf{b}_3^3 \\ & - 2880\,\mathbf{b}_2^4\mathbf{b}_3\mathbf{b}_4)\,\theta^3 + (\mathbf{b}_5^4 - 120\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5^3 \\ & + 160\,\mathbf{b}_2\mathbf{b}_4^2\mathbf{b}_5^2 + 360\,\mathbf{b}_3^2\mathbf{b}_4\mathbf{b}_5^2 \\ & - 640\,\mathbf{b}_3\mathbf{b}_4^3\mathbf{b}_5 + 256\,\mathbf{b}_4^5 - 2560\,\mathbf{b}_2^2\mathbf{b}_4^4 \\ & + 3456\,\mathbf{b}_3^5\mathbf{b}_5 + 5760\,\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4^3 \\ & + 2640\,\mathbf{b}_2^2\mathbf{b}_3^2\mathbf{b}_5^2 - 1440\,\mathbf{b}_2^3\mathbf{b}_4\mathbf{b}_5^2 - 2160\,\mathbf{b}_3^4\mathbf{b}_4^2 + 4480\,\mathbf{b}_2^2\mathbf{b}_3\mathbf{b}_4^2\mathbf{b}_5 \\ & - 10080\,\mathbf{b}_2\mathbf{b}_3^3\mathbf{b}_4\mathbf{b}_5 + 3456\,\mathbf{b}_2^5\mathbf{b}_5^2 - 3200\,\mathbf{b}_2^3\mathbf{b}_3^2\mathbf{b}_4^2 + 5120\,\mathbf{b}_2^3\mathbf{b}_3^3\mathbf{b}_5 \\ & + 6400\,\mathbf{b}_2^4\mathbf{b}_4^3 - 11520\,\mathbf{b}_2^4\mathbf{b}_3\mathbf{b}_4\mathbf{b}_5)\,\theta^4.\end{aligned}$$

Thus, for $r = 1$ we get two basic singular summands:

$$\left(\sum_{2\le k} c_s(\omega)\,\nu^{-\frac{s}{2}}\right)\exp\left(\omega\,\nu^{\frac{1}{2}}\right) \qquad \text{(for all } f \text{ of degree 1)} \tag{6.47}$$

$$\left(\nu^{-1} - \frac{1}{\omega}\,\nu^{-\frac{1}{2}}\right)\exp\left(\omega\,\nu^{\frac{1}{2}}\right) \qquad \text{(if } f(x) = f_1\,x) \tag{6.48}$$

with frequencies ω corresponding to the solutions of $\boldsymbol{\pi}_1(\theta) = 0$ *i.e.* $\boldsymbol{\pi}_1(2\,\frac{\mathbf{f}_1}{\omega^2}) = 0$ *i.e.* $\omega = (-2\,\mathbf{b}_2 f_1)^{\frac{1}{2}}$.

For $r = 2$ we have $6 = 2 \times 3$ basic summands

$$\left(\sum_{3\le k} c_s(\omega)\,\nu^{-\frac{s}{3}}\right)\exp\left(\omega\,\nu^{\frac{2}{3}}\right) \qquad \text{(for all } f \text{ of degree 2)} \tag{6.49}$$

with frequencies ω corresponding to the solutions of $\boldsymbol{\pi}_2(\theta) = 0$ *i.e.* $\boldsymbol{\pi}_2(\frac{9}{4}\frac{\mathbf{f}_2}{\omega^3}) = 0$.

For $r = 3$ we have $12 = 3 \times 4$ basic summands

$$\left(\sum_{4\leq k} c_s(\omega)\, \nu^{-\frac{s}{4}}\right) \exp\left(\omega\, \nu^{\frac{3}{4}}\right) \quad \text{(for all radial } f \text{ of degree 3)} \tag{6.50}$$

$$\left(\sum_{4\leq k} c_s(\omega)\, \nu^{-\frac{s}{4}}\right) \exp\left(\omega\, \nu^{\frac{3}{4}} + \omega_1\, \nu^{\frac{1}{4}}\right) \quad \text{(for all } f \text{ of degree 3)} \tag{6.51}$$

with main frequencies ω solution of $\boldsymbol{\pi}_3(\theta) = 0$ *i.e.* $\boldsymbol{\pi}_3(\frac{64}{27}\frac{\mathbf{f}_3}{\omega^4}) = 0$, and with secondary frequencies ω_1 dependent on the main ones and given by:

$$\omega_1 = \frac{2}{3}\frac{\mathbf{f}_1}{\omega}\frac{\left(1+(\mathbf{b}_4+3\,\mathbf{b}_2^2)\theta\right)\left(\mathbf{b}_2+(\mathbf{b}_2\mathbf{b}_4-3\,\mathbf{b}_3^2-9\,\mathbf{b}_2^3)\theta\right)}{\left(1+(\mathbf{b}_4-6\,\mathbf{b}_2^2)\theta\right)^2+9\,\mathbf{b}_2\left(2\,\mathbf{b}_3^2-\mathbf{b}_2^3\right)\theta^2} \tag{6.52}$$

$$\text{with } \mathbf{f}_1 = f_1 - \frac{1}{3}\frac{f_2^2}{f_3}, \quad \mathbf{f}_3 = f_3 \text{ and } \theta = \left(\frac{4}{3}\right)^3 \frac{\mathbf{f}_3}{\omega^4}. \tag{6.53}$$

Lastly, for $r = 4$ we have $20 = 4 \times 5$ basic summands

$$\left(\sum_{5\leq k} c_s(\omega)\, \nu^{-\frac{s}{5}}\right) \exp\left(\omega\, \nu^{\frac{4}{5}}\right) \quad \text{(for all radial } f \text{ of degree 4)} \tag{6.54}$$

$$\left(\sum_{4\leq k} c_s(\omega)\nu^{-\frac{s}{5}}\right) \exp\left(\omega\nu^{\frac{4}{5}}+\omega_2\nu^{\frac{2}{4}}+\omega_1\nu^{\frac{1}{4}}\right) \tag{6.55}$$
$$\text{(for all } f \text{ of degree 4)}$$

with main frequencies ω solution of $\boldsymbol{\pi}_4(\theta) = 0$ *i.e.* $\boldsymbol{\pi}_4(\frac{625}{256}\frac{\mathbf{f}_4}{\omega^5}) = 0$, and with secondary frequencies ω_1, ω_2 that depend on the main ones and vanish *iff* the shift-invariants $\mathbf{f}_1$ respectively $\mathbf{f}_2$ vanish.

6.5 The antipodal exchange for polynomial inputs f

As noted in the preceding subsection, the behaviour of our *nir*-transforms $h(\nu)$ at infinity in the ν-plane involves elementary exponential factors multiplied by divergent-resurgent power series

$$\varphi_\omega(\nu) = \sum_{r+1\leq k} c_s(\omega)\, \nu^{-\frac{s}{r+1}},$$

which verify simple linear ODEs easily deducible from the frequencies ω and the original ODE verified by $h(\nu)$. Therefore, to resum the $\varphi_\omega(\nu)$, which are local data at infinity, we must subject them to a formal Borel transform, which takes us back to the origin, with a new set of linear ODEs. This kicks off a resurgence ping-pong between 0 and ∞.[61] Before taking a closer look at it, let us state a useful lemma:

Lemma 6.1 (Deramification of linear homogeneous ODEs). *Let ρ be a positive integer and $\Phi(t)$ any power series in $\mathbb{C}\{t^{\frac{1}{\rho}}\}$ or $\mathbb{C}\{t^{-\frac{1}{\rho}}\}$ that verifies a linear homogeneous differential equation $P^*(t,\partial_t)\,\Phi(t) = 0$ of order δ^* and with coefficients polynomial in $t^{\frac{1}{\rho}}$ of degree d^*. Then Φ automatically verifies a new linear homogeneous differential equation $P(t,\partial_t)\,\Phi(t) = 0$ of order δ and with coefficients polynomial in t of degree d such that*

$$\delta \leq \delta^*\,\rho, \quad d \leq (1+d^*)(1+\delta^*\,(\rho-1))^2.$$

Proof. The initial, ramified differential equation, after division by the leading coefficient and deramification of the denominators, can be written uniquely in the form

$$\Phi^{(\delta^*)} = \sum_{0\leq j<\rho}\ \sum_{0\leq s<\delta^*} a_{\delta^*,j,s}\, t^{\frac{j}{\rho}}\, \Phi^{(s)} \tag{6.56}$$

with unramified coefficients $a_{\delta^*,j,s}$ that are rational in t. Under successive differentiations and eliminations of the derivatives of order larger than δ^* but $\neq i$, we then get a sequence of similar-looking equations:

$$\Phi^{(i)} = \sum_{0\leq j<\rho}\ \sum_{0\leq s<\delta^*} a_{i,j,s}\, t^{\frac{j}{\rho}}\, \Phi^{(s)} \qquad (\forall i,\, \delta^* \leq i \leq \delta^*\rho) \tag{6.57}$$

again with unramified coefficients $a_{\delta^*,j,s}$ rational in t. One then checks that there always exists a linear combination of the $(\rho-1)\,\delta^*$ equations (6.57) with coefficients $L_i(t)$ polynomial in the $a_{i',j',s'}(t)$ and therefore rational in t, that eliminates the (at most) $(\rho-1)\,\delta^*$ terms of the form $t^{\frac{j}{\rho}}$ with $1 \leq j < \rho$ and $0 \leq s < \delta^*$. After multiplication by a suitable t-polynomial, this yields the required unramified equation $P(t,\partial_t)\Phi(t) = 0$. A closer examination of the process shows that the coefficients a are of the form:

$$a_{i,j,s}(t) = \frac{b_{i,j,s}(t)}{t^{i-\delta^*}\,c(t)^{1+i-\delta^*}} \text{ with } \deg_t(c) \leq d^*,\ \deg_t(b_{i,j,s}) \leq (1+i-\delta^*)\,d^*.$$

[61] Which is quite distinct from the ping-pong between two inner generators associated with two proper base points x_i, x_j in the x-plane.

Plugging this into the elimination algorithm, we get the bound

$$\begin{aligned} d &\le (1+(\rho-1)\delta^*)\,(d^*+(\rho-1)(d^*+1)\delta^*) \quad \Longleftrightarrow \\ (1+d) &\le (1+d^*)\,(1+\delta^*\,\rho^*)^2 \qquad \text{with} \qquad \rho^* := \rho - 1 \end{aligned}$$

which, barring unlikely simplifications, is probably near-optimal. □

Let us now return to the resurgence ping-pong $0 \leftrightarrow \infty$. Graphically, we get the following sequence of transforms:

$$\begin{tikzcd}
(P_1,k_1,n_1) \arrow[dr, "12"] \arrow[rrrr, bend left] & (P_3^*,k_3,n_3) \arrow[r, "33"] & (P_3,k_3,n_3) \arrow[dr, "34"] & (P_5^*,k_5,n_5) \arrow[r, "55"] & (P_5,k_5,n_5) \\
 & (P_2,k_2,\nu_2) \arrow[u, "23"] & & (P_4,k_4,\nu_4) \arrow[u, "45"] &
\end{tikzcd}$$

Step 1. We have the polynomial-coefficient linear ODE

$$P_1(n_1, \partial_{n_1})\,k_1(n_1) = 0$$

with

$$n_1 \equiv n \sim \infty,\, k_1(n_1) \equiv k(n),\; P_1(n_1, \partial_{n_1}) \equiv P(n_1, -\partial_{n_1}).$$

Arrow 12. We perform the Borel transform from the variable $n_1 = n$ to the conjugate variable $\nu_2 = \nu$. Thus : $n_1^{-s} \mapsto \frac{\nu_2^{s-1}}{\Gamma(s)}$, $n_1 \mapsto \partial_{\nu_2}$, $\partial_{n_1} \mapsto -\nu_2$.

Step 2. We have the polynomial-coefficient linear ODE

$$P_2(\nu_2, \partial_{\nu_2})\,k_2(\nu_2) = 0$$

with

$$\nu_2 \equiv \nu \sim 0,\, k_2(\nu_2) \equiv \hat{k}(\nu),\; P_2(\nu_2, \partial_{\nu_2}) \equiv P_1(\partial_{\nu_2}, -\nu_2).$$

Arrow 23. We go from 0 to ∞, that is to say, we now solve the above ODE in powers series of negative powers of ν_2. More precisely, for an input f of degree r, we set $n_3 := \nu_2^{\frac{r}{r+1}} =: \nu_2^{\frac{1}{\kappa_3}}$, the new variable n_3 being the "critical resurgence variable" at ∞, and we then solve the ODE in negative powers of n_3.

Step 3*. We have the ramified-coefficient linear ODE

$$P_3^*(n_3, \partial_{n_3})\,k_3(n_3) = 0$$

with

$$n_3^{\kappa_3} \equiv \nu_2 \;\text{ but } n_3 \sim \infty\,,\; k_3(n_3) \equiv k_2(\nu_2),\; P_3^*(n_3, \partial_{n_3})$$
$$\equiv P_2\left(n_3^{\kappa_3}, \frac{n_3^{\kappa_3-1}}{\kappa_3}\partial_{n_3}\right).$$

Arrow 33. Since for an input f of degree r, we must take $\kappa_3 = \frac{r+1}{r}$, this leads to a ramification of order r in the coefficients of P_3^*. We then apply the above Lemma 6.1 with $\rho = r$ to deramify P_3^* to P_3.

Step 3. We have the polynomial-coefficient linear ODE

$$P_3(n_3, \partial_{n_3})\, k_3(n_3) = 0$$

with n_3 and k_3 as in Step 3* but with a linear homogeneous differential operator P_3 which, unlike P_3^*, is polynomial in n_3.

Arrow 34. We perform the Borel transform from the variable n_3 to the conjugate variable ν_4. Thus: $n_3^{-s} \mapsto \frac{\nu_4^{s-1}}{\Gamma(s)}$, $n_3 \mapsto \partial_{\nu_4}$, $\partial_{n_3} \mapsto -\nu_4$.

Step 4. We have the polynomial-coefficient linear ODE

$$P_4(\nu_4, \partial_{\nu_4})\, k_4(\nu_4) = 0$$

with ν_4 conjugate to n_3 and $P_4(\nu_4, \partial_{\nu_4}) \equiv P_3(\partial_{\nu_4}, -\nu_4)$.

Arrow 45. We go from 0 to ∞ and from increasing power series of the variable ν_4 to decreasing power series of the variable n_5. For an input f of degree r, we set $n_5 := \nu_4^{\frac{r+1}{r}} =: \nu_4^{\frac{1}{\kappa_5}} = \nu_4^{\kappa_3}$, the new variable n_5 being the "critical resurgence variable" at ∞.

Step 5*. We have the ramified-coefficient linear ODE

$$P_5^*(n_5, \partial_{n_5})\, k_5(n_5) = 0$$

with

$$n_5^{\kappa_5} \equiv \nu_4 \;\text{ but } n_5 \sim \infty\,,\; k_5(n_5) \equiv k_4(\nu_4),\; P_5^*(n_5, \partial_{n_5})$$
$$\equiv P_4\left(n_5^{\kappa_5}, \frac{n_5^{\kappa_5-1}}{\kappa_5}\partial_{n_5}\right).$$

Arrow 55. Since for an input f of degree r, we must take $\kappa_5 = \frac{r}{r+1} = \frac{1}{\kappa_3}$, this leads to a ramification of order $r+1$ in the coefficients of P_5^*. We then apply once again the above Lemma 6.1 with $\rho = r + 1$ to deramify P_5^* to P_5.

Step 5. We have the polynomial-coefficient linear ODE

$$P_5(n_5, \partial_{n_5})\, k_5(n_5) = 0$$

with n_5 and k_5 as in step 5* but with a linear homogeneous differential operator P_5 which, unlike P_5^*, is polynomial in n_5.

6.6 ODEs for monomial inputs F

General meromorphic inputs F, with more than one zero or pole, shall be investigated in Section 7.2 and Section 8.3-4 with the usual *nir-mir* approach. Here, we shall restrict ourselves to strictly *monomial* F, *i.e.* with only one zero *or* pole (but of abitrary order p), for these monomial inputs, and only they, give rise to *nir* transforms that verify linear ODEs with polynomial coefficients. So for now our inputs shall be:

$$f(x) := +p\log(1+p\,x),\;\; F(x) := (1+p\,x)^{-p} \quad (p \in \mathbb{N}^*) \tag{6.58}$$
$$f(x) := -p\log(1-p\,x),\;\; F(x) := (1-p\,x)^{+p} \quad (p \in \mathbb{N}^*) \tag{6.59}$$

and we shall set as usual:

$$\begin{aligned}
k(n) &:= \text{singular}\left(\int_0^{\infty} e_{\#}^{-\beta(\partial_\tau)\, f(\frac{\tau}{n})}\; d\tau\right) && \in \Gamma(1/2)\, n^{1/2}\, \mathbb{Q}[[n^{-1}]]\\
\overset{\frown}{h}\,(\nu) &:= \text{formal}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\, e^{\nu\, n}\frac{dn}{n}\right) = h(\nu) && \in \nu^{-1/2}\, \mathbb{Q}\{\nu\}\\
\hat{k}\,(\nu) &:= \text{formal}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\, e^{\nu\, n} dn\right) = h'(\nu) && \in \nu^{-3/2}\, \mathbb{Q}\{\nu\}.
\end{aligned}$$

Unlike with the polynomial inputs f of Section 6.2-5, the global *nir* transforms now verify no (variable) polynomial linear-homogeneous ODEs. Only their singular parts, which in the present case ($\forall p$) always consist of semi-entire powers of the variable, do verify (covariant) linear ODEs with polynomial coefficients. These equations depend only on the absolute value $|p|$ and read:

$$\begin{aligned}
\left(n + n\partial_n - \frac{|p|}{2}\right)^{|p|} k(n) &= n^{|p|}\, k(n)\\
\left(\partial_\nu - \nu\,\partial_\nu - \frac{|p|}{2}\right)^{|p|} h(\nu) &= (\partial_\nu)^{|p|}\, h(\nu)\\
\left(\partial_\nu - \nu\,\partial_\nu - 1 - \frac{|p|}{2}\right)^{|p|} \hat{k}(\nu) &= (\partial_\nu)^{|p|}\, \hat{k}(\nu).
\end{aligned}$$

If we regard n and ν no longer as commutative variables (as in Section 4 and Section 5), but as non-commutative ones bound by $[n,\nu]=1$ (as in the preceding sections), our covariant ODEs read:

$$P(n,-\partial_n)\,k(n)=0,\quad \partial_\nu^{-1}\,P(\partial_\nu,\nu)\,\partial_\nu\,h(\nu)=0,\quad P(\partial_\nu,\nu)\,\hat{k}(\nu)=0$$
$$\text{with}\quad P(n,\nu):=\left(n-n\,\nu-\frac{|p|}{2}\right)^{|p|}-n^{|p|}=\left(n-\nu\,n-1-\frac{|p|}{2}\right)^{|p|}-n^{|p|}.$$

If we now apply the covariance relation (6.20) to the shifts (ϵ,η):

$$\epsilon:=-1/|p|\;,\;\;\eta:=\int_0^\epsilon f(x)\,dx=1\;,\;\;{}^{\epsilon}f(x)=|p|\,\log(|p|x)$$

we find a centered polynomial P_* predictably simpler than P:

$$P_*(n,\nu)=P(n,\nu+\eta)=\left(-n\,\nu-\frac{|p|}{2}\right)^{|p|}-n^{|p|}.$$

Although our covariant operators $P(n,\nu)$ are now much simpler, and of far lower degree in n, than was the case for polynomial inputs f, their form is actually harder to derive. As for their dependence on $|p|$ rather than p, it follows from the general parity relation for the nir transform (*cf.* Section 4.10), but here it also makes direct formal sense. Indeed, in view of $[n,\nu]=1$, we have the chain of formal equivalences:

$$\left\{\left(n-n\,\nu-\frac{p}{2}\right)^{p}\,k(n)=n^p\,k(n)\right\}\Longleftrightarrow$$
$$\left\{k(n)=\left(n-n\,\nu-\frac{p}{2}\right)^{-p}\,n^p\,k(n)\right\}\Longleftrightarrow$$
$$\left\{k(n)=n^p\,\left(n-n\,\nu+\frac{p}{2}\right)^{-p}\,k(n)\right\}\Longleftrightarrow$$
$$\left\{n^{-p}\,k(n)=\left(n-n\,\nu+\frac{p}{2}\right)^{-p}\,k(n)\right\}$$

which reflects the invariance of $P(n,\nu)$[62] under the change $p\mapsto -p$.

From the form of the centered differential operator, it is clear that $h(1-\nu)$ has all its irregular singular points over the unit roots, plus a regular singular point at infinity.

Remark. Although both inputs $f_1(x)=\frac{1}{p}\,x^p-1$ and $f_2(x)=\pm p\log(1\pm p\,x)$ lead to *nir*-transforms $h_1(1-\nu)$ and $h_2(1-\nu)$ with radial symmetry

[62] Or more accurately: the invariance of the relation $P(n,\nu)\,k(n)=0$.

and singular points over the unit roots of order p, there are far-going differences:

(i) h_2 verifies much simpler ODEs than h_1;
(ii) conversely, h_1 verifies much simpler resurgence equations than h_2 (see *infra*);
(iii) the singularities of h_1 over ∞ are of divergent-resurgent type (see Section 6.4-5) whereas those of h_2 are merely ramified-convergent (see Section 6.7).

Let us now revert to our input (6.58) or (6.59) with the corresponding *nir* transform $h(\nu)$ and its linear ODE. That ODE always has very explicit power series solutions at $\nu = 0$ and $\nu = \infty$ and, as we shall see, this is what really matters. At $\nu = 0$ the solutions are of the form:

$$k(n) = \sum_{s \in -\frac{1}{2}+\mathbb{N}} k_s\, n^{-s}, \quad h(\nu) = \sum_{s \in -\frac{1}{2}+\mathbb{N}} k_s\, \nu^s \qquad \text{(relevant)}$$

$$k^{\text{en}}(n) = \sum_{s \in \mathbb{N}} k_s\, n^{-s}, \qquad h^{\text{en}}(\nu) = \sum_{s \in \mathbb{N}} k_s\, \nu^s \qquad \text{(irrelevant)}$$

but only for $p \in \{\pm 1, \pm 2, \pm 3\}$ are the coefficients explicitable.

The case $p = \pm 1$.

$$k_{-\frac{1}{2}+r} = 0 \quad \text{if} \quad r \geq 1 \quad \text{and} \quad k_{-\frac{1}{2}} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$$

$$h_{-\frac{1}{2}+r} = 0 \quad \text{if} \quad r \geq 1 \quad \text{and} \quad k_{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}.$$

The case $p = \pm 2$.

$$k_{-\frac{1}{2}+r} = 2^{-5r}\,\frac{(2r)!(2r)!}{r!r!r!}\,k_{-\frac{1}{2}} \quad \text{with} \quad k_{-\frac{1}{2}} = \left(\frac{\pi}{8}\right)^{\frac{1}{2}}$$

$$h_{-\frac{1}{2}+r} = 2^{-3r}\,\frac{(2r)!}{r!r!}\,h_{-\frac{1}{2}} \quad \text{with} \quad h_{-\frac{1}{2}} = \left(\frac{1}{8}\right)^{\frac{1}{2}}.$$

The case $p = \pm 3$. The coefficients of k, h have no simple multiplicative structure, but the entire analogues k^{en}, h^{en} are simple superpositions of hypergeometric series.

6.7 Monomial inputs F: global resurgence

Let us replace the pair (h, P) by (ho, Po) with

$$\begin{aligned} ho(\nu) &:= h(1-\nu); \\ Po(n,\nu) &:= (-1)^p\, P(-n, 1-\nu) = \left(\nu\, n + \frac{p}{2}\right)^p - n^p \end{aligned} \tag{6.60}$$

so as to respect the radial symmetry and deal with a function ho having all its singular points over the unit roots $e_j = \exp(2\pi i j/p)$. At the crucial points $\nu_0 \in \{0, \infty, e_0, \ldots, e_{p-1}\}$ the p-dimensional kernel of the operator

$$\mathcal{P}o(\partial, \nu + \nu_0) = \left((\nu + \nu_0)\partial_\nu + \frac{p}{2}\right)^p - \left(\partial_\nu\right)^p \tag{6.61}$$

is spanned by the following systems of fundamental solutions

$$\begin{aligned}
&\text{at } \nu_0 = 0 : && ho_s(\nu) \in \nu^s\, \mathbb{C}\{\nu^p\} && (0 \le s \le p-1)\\
&\text{at } \nu_0 = \infty : && hi_s(\nu) \in \bigoplus_{0 \le \sigma \le s-1} \left(\nu^{-p/2}\, \mathbb{C}\{\nu^{-1}\} \frac{(\log \nu)^\sigma}{\sigma!}\right) && (0 \le s \le p-1)\\
&\text{at } \nu_0 = e_j : && ha_j(\nu) \in \nu^{-1/2}\, \mathbb{C}\{\nu\} && \\
& && ha_{j,s}(\nu) \in \mathbb{C}\{\nu\} && (1 \le s \le p-1).
\end{aligned}$$

The singular solutions ha_j (normalised in a manner consistent with the radial symmetry) are, up to sign, none other than the *inner generators* whose resurgence properties we want to describe. For $p \ge 3$, their coefficients have no transparent expression, but the coefficients of the hi_s and, even more so, those of the ho_s do possess a very simple multiplicative structure, which allows us to apply the method of coefficient asymptotics in Section 2.3 to derive the resurgence properties of the ha_j, and that too from “both sides” — from 0 and ∞. A complete treatment shall be given in [12] but here we shall only state the result and describe the closed resurgence system governing the behaviour of the ha_j. To that end, we consider their Laplace integrals along any given axis $\arg \nu = \theta$, with the “location factor” $e^{-e_j n}$:

$$haa_j^\theta(n) := e^{-e_j n} \int_0^{e^{i\theta}\infty} ha_j(\nu)\, d\nu \qquad (\theta \in \mathbb{R},\ j \in \mathbb{Z}/p\mathbb{Z}). \tag{6.62}$$

Everything boils down to describing the effect on the system $\{haa_1^\theta, \ldots, haa_p^\theta\}$ of crossing a singular axis $\theta_0 = \arg(e_{j_2} - e_{j_1})$, *i.e.* of going from $\theta_0 - \epsilon$ to $\theta_0 + \epsilon$. The underlying ODE being linear, such a crossing will simply subject $\{haa_1^\theta, \ldots, haa_p^\theta\}$ to a linear transformation with constant coefficients. Moreover, since all $ha_j(\nu)$ are in $\nu^{-1/2}\,\mathbb{C}\{\nu\}$, two full turns (*i.e.* changing θ to $\theta + 4\pi$) ought to leave $\{haa_1^\theta, \ldots, haa_p^\theta\}$ unchanged. All the above facts can be derived in a rather straightforward manner by resurgence analysis (see [12]) but, when translated into matrix algebra, they lead to rather complex matrices and to remarkable, highly non-trivial relations between these. Of course, the relations in question also admit *“direct” algebraic* proofs, but these are rather difficult – and in any case much longer than their *"indirect" analytic* derivation. The long subsection which follows is entirely devoted to this "algebraic" description of the resurgence properties of the ha_j.

6.8 Monomial inputs F: algebraic aspects

Some elementary matrices

Eventually, ϵ will stand for -1 and ϵ^q for $e^{\pi i q}$, $\forall q \in \mathbb{Q}$, but for greater clarity ϵ shall be kept free (unassigned) for a while. We shall encounter both ϵ-carrying matrices, which we shall underline, and ϵ-free matrices. For each p, we shall also require the following elementary square matrices ($p \times p$):

- $\mathcal{I}$: identity;
- $\underline{\mathcal{I}}$: ϵ-carrying diagonal;
- $\mathcal{J}$: Jordan correction;
- $\mathcal{P}$: unit shift;
- $\mathcal{Q}$: twisted unit shift.

These are hollow matrices, whose only nonzero entries are:

$$
\begin{array}{llll}
\underline{\mathcal{I}}[i,j] = \epsilon^{j/p} \ \textit{if} \ j=i & & & \\
\mathcal{J}[i,j] = \ 1 \ \text{if} \ j=i+1 & & & \\
\mathcal{P}[i,j] = \ 1 \ \text{if} \ j=i+1 \mod p & & \mathcal{P}^k[i,j] = \ 1 \ \text{if} \ j=i+k \mod p \\
\mathcal{Q}[i,j] = \ 1 \ \text{if} \ j=i+1 & & \mathcal{Q}^k[i,j] = \ 1 \ \text{if} \ j=i+k \\
\mathcal{Q}[i,j] = -1 \ \text{if} \ j=i+1-p & & \mathcal{Q}^k[i,j] = -1 \ \text{if} \ j=i+k-p.
\end{array}
$$

The simple-crossing matrices $\underline{\mathcal{M}}_k$, $\mathcal{M}_k$

Let *fr*(x) respectively *en*(x) denote the *fractional* respectively *entire* part of any real x:

$$x \equiv \mathrm{fr}(x) + \mathrm{en}(x) \quad \textit{with} \quad x \in \mathbb{R} \ , \quad \mathrm{fr}(x) \in [0, 1[\ , \quad \mathrm{en}(x) \in \mathbb{Z}.$$

Fix $p \in \mathbb{N}^*$ and set $e_j := \exp(2\pi i j/p), \forall j \in \mathbb{Z}$. For any $k \in \frac{1}{2}\mathbb{Z}$, it is convenient to denote θ_k the axis of direction $2\pi(\frac{k}{p} + \frac{3}{4})$, *i.e.* the axis from e_{j_1} to e_{j_2} for any pair $j_1, j_2 \in \mathbb{Z}$ such that $j_1 + j_2 = 2k \mod p$ and $(k < j_1 < j_2)_p^{\mathrm{circ}}$. The matrix $\underline{\mathcal{M}}_k$ corresponding to the (counterclockwise) crossing of the axis θ_k has the following elementary entries:

$$
\begin{aligned}
\underline{\mathcal{M}}_k[i,\, j] =& 1 && \text{if} \quad i = j\\
\underline{\mathcal{M}}_k[i,\, j] =& -\, \epsilon^{\mathrm{fr}(\frac{j-k}{p})-\mathrm{fr}(\frac{i-k}{p})} \frac{p!}{(|i-j|)!(p-|i-j|)!} && \\
& && \text{if} \quad \mathrm{fr}\left(\frac{i+j-2k}{p}\right) = 0\\
& && \text{and} \quad \mathrm{fr}\left(\frac{i-k}{p}\right) > \mathrm{fr}\left(\frac{j-k}{p}\right)\\
\underline{\mathcal{M}}_k[i,\, j] =& 0 && \text{otherwise.}
\end{aligned}
$$

Alternatively, we may start from the simpler matrix $\underline{\mathcal{M}}_0$:

$$
\begin{aligned}
\underline{\mathcal{M}}_0[i,\, j] &= 1 && \text{if} \quad i = j\\
\underline{\mathcal{M}}_0[i,\, j] &= -\epsilon^{\frac{j-i}{p}} \frac{p!}{(i-j)!(p-i+j)!} && \text{if} \quad i > j \text{ and } i+j = 2k \mod p\\
\underline{\mathcal{M}}_0[i,\, j] &= 0 && \text{otherwise}
\end{aligned}
$$

and deduce the general $\underline{\mathcal{M}}_k$ under the rules:

$$\underline{\mathcal{M}}_k[i,\, j] = \underline{\mathcal{M}}_0[[i-k]_p, [j-k]_p] \qquad \text{with} \quad [x]_p := p.\,\mathrm{en}\left(\frac{x}{p}\right)$$

$\underline{\mathcal{M}}_k$ carries unit roots of order $2p$ (hence the underlining) but can be turned into a unit root-free matrix $\mathcal{M}_k$ under a k-independent conjugation:

$$\mathcal{M}_k = \underline{\mathcal{I}}\ \ \underline{\mathcal{M}}\ \ \underline{\mathcal{I}}^{-1} \tag{6.63}$$

with the elementary diagonal matrix $\underline{\mathcal{I}}$ defined above. We may therefore work with the simpler matrices $\mathcal{M}_k$ whose entries are:

$$
\begin{aligned}
\mathcal{M}_k[i,\, j] =& 1 && \text{if} \quad i = j\\
\mathcal{M}_k[i,\, j] =& -\, \epsilon^{\mathrm{en}(\frac{j-k}{p})-\mathrm{en}(\frac{i-k}{p})} \frac{p!}{(|i-j|)!(p-|i-j|)!} && \\
& && \text{if} \quad \mathrm{fr}\left(\frac{i+j-2k}{p}\right) = 0\\
& && \text{and} \quad \mathrm{fr}\left(\frac{i-k}{p}\right) > \mathrm{fr}\left(\frac{j-k}{p}\right)\\
\mathcal{M}_k[i,\, j] =& 0 && \text{otherwise.}
\end{aligned}
$$

However, since $\underline{\mathcal{I}}$ and $\mathcal{P}$ do not commute, we go from $\underline{\mathcal{M}}_k$ to $\underline{\mathcal{M}}_{k+1}$ under the *regular* shift $\mathcal{P}$ but from $\mathcal{M}_k$ to $\mathcal{M}_{k+1}$ under the *twisted* shift $\mathcal{Q}$:

$$\underline{\mathcal{M}}_{k+1} = \mathcal{P}^{-1}\ \underline{\mathcal{M}}_k\ \mathcal{P}, \quad \mathcal{M}_{k+1} = \mathcal{Q}^{-1}\ \mathcal{M}_k\ \mathcal{Q}. \tag{6.64}$$

The multiple-crossing matrices $\underline{\mathcal{M}}_{k_2,k_1}$, $\mathcal{M}_{k_2,k_1}$
For any $k_1, k_2 \in \frac{1}{2}\mathbb{Z}$ such that $k_2 > k_1$ we set :

$$\underline{\mathcal{M}}_{k_2,k_1} := \underline{\mathcal{M}}_{k_2}\ \underline{\mathcal{M}}_{k_2-\frac{1}{2}}\ \underline{\mathcal{M}}_{k_2-\frac{2}{2}}\ \dots\ \underline{\mathcal{M}}_{k_1+\frac{3}{2}}\ \underline{\mathcal{M}}_{k_1+\frac{2}{2}}\ \underline{\mathcal{M}}_{k_1+\frac{1}{2}} \tag{6.65}$$

$$\mathcal{M}_{k_2,k_1} := \mathcal{M}_{k_2}\ \mathcal{M}_{k_2-\frac{1}{2}}\ \mathcal{M}_{k_2-\frac{2}{2}}\ \dots\ \mathcal{M}_{k_1+\frac{3}{2}}\ \mathcal{M}_{k_1+\frac{2}{2}}\ \mathcal{M}_{k_1+\frac{1}{2}}. \tag{6.66}$$

For $k_2 < k_1$ or $k_2 = k_1$ we set of course:

$$\underline{\mathcal{M}}_{k_2,k_1} := \underline{\mathcal{M}}^{-1}_{k_1,k_2}, \quad \mathcal{M}_{k_2,k_1} := \mathcal{M}^{-1}_{k_1,k_2}, \quad \underline{\mathcal{M}}_{k,k} := \mathcal{M}_{k,k} := \mathcal{I}$$

thus ensuring the composition rule:

$$\underline{\mathcal{M}}_{k_3,k_2}\ \underline{\mathcal{M}}_{k_2,k_1} = \underline{\mathcal{M}}_{k_3,k_1}, \quad \mathcal{M}_{k_3,k_2}\ \mathcal{M}_{k_2,k_1} = \mathcal{M}_{k_3,k_1} \quad \left(\forall k_i \in \frac{1}{2}\mathbb{Z}\right).$$

Since $\underline{\mathcal{M}}_{p+k} \equiv \underline{\mathcal{M}}_k$ and $\mathcal{M}_{p+k} \equiv \mathcal{M}_k$ for all k (p-periodicity), each full-turn matrix $\underline{\mathcal{M}}_{p+k,k}$ or $\mathcal{M}_{p+k,k}$ is conjugate to any other. It turns out, however, that just two of them (corresponding to $k \in \{0, 1\}$ if $p = 0$ or $1 \mod 4$, and to $k \in \{\pm\frac{1}{2}\}$ if $p = 2$ or $3 \mod 4$) admit a simple or at least tolerably explicit normalisation (*i.e.* a conjugation to the canonical Jordan form, or in this case, a more convenient variant thereof). That normalisation involves remarkable lower diagonal matrices $\mathcal{L}$ and $\mathcal{R}$. To construct $\mathcal{L}$ and $\mathcal{R}$, however, we require a set of rather intricate polynomials H_d^δ.

The auxiliary polynomials $H_d^\delta(x, y)$
These polynomials, of global degree d in each of their two variables x, y, also depend on an integer-valued parameter $\delta \in \mathbb{Z}$. They are d-inductively determined by the following system of difference equations in y, along with the initial conditions for $y = 0$:

$$H_d^\delta(x, y) = H_d^\delta(x, y-1) + (x-d)\, H_{d-1}^\delta(x, y-1) \tag{6.67}$$

$$H_d^\delta(x, 0) = \frac{(x+\delta+d)!}{(x+\delta)!} = \prod_{0<d_1\le d} (x+\delta+d_1). \tag{6.68}$$

One readily sees that this induction leads to the direct expression:

$$H_d^\delta(x, y) = \sum_{d_1=0}^{d} \frac{(x-1-d+d_1)!}{(x-1-d)!} \frac{(x+\delta+d-d_1)!}{(x+\delta)!} \frac{y!}{d_1!(y-d_1)!} \tag{6.69}$$

$$= \sum_{d_1=0}^{d} \frac{1}{d_1!} \prod_{0\le k_1<d_1} (x-d+k_1) \prod_{1\le k_2\le d-d_1} (x+\delta+k_2) \prod_{0\le k_3<d_1} (y-k_3)$$

which is turn can be shown to be equivalent to:

$$H_d^{\delta}(x,y)=\sum_{d_1=0}^{d}\left[\left[\frac{\delta+2d_1}{\delta+d_1}\right]\right]!!\frac{(2d+d_1-y)!}{(d+2d_1-y)!}\prod_{0\le d_2\le d}^{d_2\ne d_1}\frac{(x-d_2)}{(d_1-d_2)} \tag{6.70}$$

$$=\sum_{d_1=0}^{d}\left[\left[\frac{\delta+2d_1}{\delta+d_1}\right]\right]!!\prod_{d_1<d_3\le d}(d+d_1+d_3-y)\prod_{0\le d_2\le d}^{d_2\ne d_1}\frac{(x-d_2)}{(d_1-d_2)} \tag{6.71}$$

with:

$$\begin{aligned}\left[\left[\frac{a}{b}\right]\right]!! &:= \frac{a!}{b!} && \text{if} \quad a,b\in\mathbb{N}\\ &:= (-1)^{a-b}\frac{(-1-b)!}{(-1-a)!} && \text{if} \quad a,b\in-\mathbb{N}^* \\ &:= 0 && \text{otherwise.}\end{aligned} \tag{6.72}$$

The left normalising matrix $\mathcal{L}$

$$\begin{array}{lll}
\text{if } i<j & (\forall p) & :\mathcal{L}[i,j]=0\\
\text{if } p=0 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p, & i+j\le p+2 & :\mathcal{L}[i,j]=(-1)^i\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p, & i+j> p+2 & :\mathcal{L}[i,j]=(-1)^j\frac{(i-1)!}{(2j-3)!(p-2j+2)!}H^1_{p-i}(j-2,p)\\
2j>p & & :\mathcal{L}[i,j]=(-1)^j\frac{(p-j)!}{(p-i)!(i-j)!}\\
\text{if } p=1 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p+1, & i+j\le p+2 & :\mathcal{L}[i,j]=(-1)^i\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p+1, & i+j> p+2 & :\mathcal{L}[i,j]=-(-1)^j\frac{(i-1)!}{(2j-3)!(p-2j+2)!}H^1_{p-i}(j-2,p)\\
2j>p+1 & & :\mathcal{L}[i,j]=(-1)^j\frac{(p-j)!}{(p-i)!(i-j)!}\\
\text{if } p=2 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p, & i+j\le p+1 & :\mathcal{L}[i,j]=(-1)^i\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p, & i+j> p+1 & :\mathcal{L}[i,j]=-(-1)^j\frac{(i-1)!}{(2j-2)!(p-2j+1)!}H^0_{p-i}(j-1,p)\\
2j>p & & :\mathcal{L}[i,j]=(-1)^j\frac{(p-j)!}{(p-i)!(i-j)!}\\
\text{if } p=3 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p-1, & i+j\le p+1 & :\mathcal{L}[i,j]=(-1)^{i-1}\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p-1, & i+j> p+1 & :\mathcal{L}[i,j]=(-1)^{i-1}\frac{(i-1)!}{(2j-2)!(p-2j+1)!}H^0_{p-i}(j-1,p)\\
2j>p-1 & & :\mathcal{L}[i,j]=(-1)^{p-j}\frac{(p-j)!}{(p-i)!(i-j)!}.
\end{array}$$

The right normalising matrix $\mathcal{L}$

$$
\begin{array}{lll}
\text{if } \ i<j & (\forall p) & :\mathcal{R}[i,j]=0\\[4pt]
\text{if } \ p=0 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p-2,\ i+j\le p & & :\mathcal{R}[i,j]=(-1)^{i}\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p-2,\ i+j> p & & :\mathcal{R}[i,j]=(-1)^{j}\frac{(i-1)!}{(2j-1)!(p-2j)!}\,H^{-1}_{p-i}(j,p)\\
2j> p-2 & & :\mathcal{R}[i,j]=(-1)^{j}\frac{(p-j)!}{(p-i)!(i-j)!}\\[4pt]
\text{if } \ p=1 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p-1,\ i+j\le p & & :\mathcal{R}[i,j]=(-1)^{i}\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p-1,\ i+j> p & & :\mathcal{R}[i,j]=-(-1)^{j}\frac{(i-1)!}{(2j-1)!(p-2j)!}\,H^{-1}_{p-i}(j,p)\\
2j> p-1 & & :\mathcal{R}[i,j]=(-1)^{j}\frac{(p-j)!}{(p-i)!(i-j)!}\\[4pt]
\text{if } \ p=2 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p-2,\ i+j\le p-1 & & :\mathcal{R}[i,j]=(-1)^{i}\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p-2,\ i+j> p-1 & & :\mathcal{R}[i,j]=-(-1)^{j}\frac{(i-1)!}{(2j)!(p-2j-1)!}\,H^{-2}_{p-i}(j+1,p)\\
2j> p-2 & & :\mathcal{R}[i,j]=(-1)^{j}\frac{(p-j)!}{(p-i)!(i-j)!}\\[4pt]
\text{if } \ p=3 & \text{mod } 4 \quad \text{and} \ldots & \\
2j\le p-3,\ i+j\le p-1 & & :\mathcal{R}[i,j]=(-1)^{i-1}\frac{(i-1)!}{(j-1)!(i-j)!}\\
2j\le p-3,\ i+j> p-1 & & :\mathcal{R}[i,j]=(-1)^{i-1}\frac{(i-1)!}{(2j)!(p-2j-1)!}\,H^{-2}_{p-i}(j+1,p)\\
2j> p-3 & & :\mathcal{R}[i,j]=(-1)^{p-j}\frac{(p-j)!}{(p-i)!(i-j)!}.
\end{array}
$$

Normalisation identities for the full-turn matrices $\mathcal{M}_{p+k,k}$

$$
\begin{array}{rcll}
\mathcal{L}\ \mathcal{M}_{p+1,1}\ \mathcal{L}^{-1} &=& (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \text{if } \ p=0 \text{ or } 1 \quad \text{mod } 4\\
\mathcal{R}\ \mathcal{M}_{p+0,0}\ \mathcal{R}^{-1} &=& (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \text{if } \ p=0 \text{ or } 1 \quad \text{mod } 4\\[6pt]
\mathcal{L}\ \mathcal{M}_{p+\frac{1}{2},+\frac{1}{2}}\ \mathcal{L}^{-1} &=& (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \text{if } \ p=2 \text{ or } 3 \quad \text{mod } 4\\
\mathcal{R}\ \mathcal{M}_{p-\frac{1}{2},-\frac{1}{2}}\ \mathcal{R}^{-1} &=& (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \text{if } \ p=2 \text{ or } 3 \quad \text{mod } 4
\end{array}
$$

with $\mathcal{I}$ denoting the identity matrix and $\mathcal{J}$ the matrix carrying a maximal upper-Jordan side-diagonal:

$$\mathcal{J}[i,j]=1 \quad \text{if} \quad j=1+i \quad \text{and} \quad \mathcal{J}[i,j]=0 \quad \text{otherwise.}$$

This result obviously implies that *all* full-rotation matrices $\mathcal{M}_{p+k,\,k}$ are also conjugate to $(-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p}$ but the point, as already mentioned, is that only for $k\in\{0,1\}$ or $\{\pm\frac{1}{2}\}$ do we get an explicit conjugation with simple, lower-diagonal matrices like $\mathcal{L}, \mathcal{R}$. As for the choice of $(-1)^{p-1}(\mathcal{I}+\mathcal{J})^p$ rather than $(-1)^{p-1}\mathcal{I}+\mathcal{J}$ as normal form, it is simply a matter of convenience, and a further, quite elementary conjugation, immediately takes us from the one to the other.

Defining identities for the normalising matrices $\mathcal{L}, \mathcal{R}$

$$\mathcal{R} = (\mathcal{I}+\mathcal{J})\,\mathcal{L}\,\mathcal{Q}^{-1} \tag{6.73}$$
$$\mathcal{R} = \mathcal{L}\,\mathcal{W} \tag{6.74}$$

with the twisted shift matrix $\mathcal{Q}$ defined right at the beginning of Section 6.8 and with

$$\begin{array}{lll} \mathcal{W} = \mathcal{M}_{1,0} & \text{if } p = 0 \text{ or } 1 & \mod 4 \\ \mathcal{W} = \mathcal{M}_{\frac{1}{2},-\frac{1}{2}} & \text{if } p = 2 \text{ or } 3 & \mod 4. \end{array}$$

The matrix entries of $\mathcal{W}$ are elementary binomial coefficients:

$$\begin{array}{ll} \text{if } i<j & : \mathcal{W}[i,j] = 0 \\ \text{if } i=j & : \mathcal{W}[i,j] = 1 \\ \text{if } i>j \text{ and} \ldots & \\ p\in\{0,1\} \bmod 4 \text{ and } p-i-j\in\{1,2\} & : \mathcal{W}[i,j] = -\dfrac{p!}{(i-j)!(p-i+j)!} \\ p\in\{2,3\} \bmod 4 \text{ and } p-i-j\in\{0,1\} & : \mathcal{W}[i,j] = -\dfrac{p!}{(i-j)!(p-i+j)!} \\ \text{otherwise} & : \mathcal{W}[i,j] = 0. \end{array}$$

If we now eliminate either $\mathcal{R}$ (respectively $\mathcal{L}$) from the system (6.73), (6.74) and express the remaining matrix as a sum of an *elementary part* (which corresponds to the two extreme subdiagonal zones and carries only binomial entries) and a *complex part* (which corresponds to the middle subdiagonal zone and involves the intricate polynomials H_d^δ), we get a linear system which, as it turns out, completely determines $\mathcal{L}^{\text{comp.}}$ or $\mathcal{R}^{\text{comp.}}$ (viewed as unknown) in terms $\mathcal{L}^{\text{elem.}}$ or $\mathcal{R}^{\text{elem.}}$ (viewed as known). Thus:

$$(\mathcal{I}+\mathcal{J})\,(\mathcal{L}^{\text{elem.}}+\mathcal{L}^{\text{comp.}}) = (\mathcal{L}^{\text{elem.}}+\mathcal{L}^{\text{comp.}})\,\mathcal{W}\,\mathcal{Q}. \tag{6.75}$$

To understand just how special the value $\epsilon = -1$ and the case of full-rotation matrices are, let us briefly examine, first, the case of full rotations with unassigned ϵ, then the case of partial rotations with $\epsilon = -1$.

Complement: full rotations with $\epsilon \neq -1$
Keeping ϵ free and setting $V_p(t,\epsilon) := \det(t\,\mathcal{I} - \mathcal{M}_{p,0})$ we get:

$$\begin{aligned}
V_2(t,\epsilon) &= (t+1)^2 \; - 2^2 \; (1+\epsilon)\,t \\
V_3(t,\epsilon) &= (t-1)^3 \; + 3^3 \; (1+\epsilon)\,t\,\epsilon \\
V_4(t,\epsilon) &= (t+1)^4 \; - 2^3 \; (1+\epsilon)\,t\,(1+16\,\epsilon+32\,\epsilon^2+14\epsilon\,t+t^2) \\
V_5(t,\epsilon) &= (t-1)^5 \; + 5^4 \; (1+\epsilon)\,t\,\epsilon\,(1+5\,\epsilon+5\,\epsilon^2+3\,t+t^2).
\end{aligned}$$

Etc... The only conspicuous properties of the V_p polynomials seem to be:

$$V_p(t,-1) = \big(t+(-1)^p\big)^p \tag{6.76}$$

$$V_p(1,\epsilon) = \epsilon^p \; V_p(1,\epsilon^{-1}) \tag{6.77}$$

(6.76) follows from the short analysis argument given in Section 6.7, and we have devoted the bulk of the present Section (Section 6.8) to checking it algebraically. As for the self-inversion property (6.77), it directly follows from the way the simple-crossing matrices $\mathcal{M}_k$ are constructed. As far as we can see, the V_p polynomials appear to possess only one additional property, albeit a curious one (we noticed it empirically and didn't attempt a proof). It is this: for p prime ≥ 5 and $t = 1$ we have (at least up to $p = 59$):

$$V_p(1,\epsilon) = \det(\mathcal{I} - \mathcal{M}_{p,0}) = p^p\,\epsilon\,(1+\epsilon)\,(1+\epsilon+\epsilon^2)^{\kappa(p)}\;W_p(\epsilon) \tag{6.78}$$

with $\kappa(p) = 1$ (respectively 2) if $p = 2$ (respectively 1) $\mod 4$ and some $\mathbb{Q}$-irreducible polynomial $W(\epsilon) \in \mathbb{Z}(\epsilon)$. However, $V_p(t,\epsilon) \neq 0 \mod 1+\epsilon+\epsilon^2$, which reduces the above relation (6.78) to a mere oddity. [63] More generally, the "semi-periodicity" in k of $\mathcal{M}_{k,0}$ that we noticed for $\epsilon = -1$ has no counterpart for any other value of ϵ, not even for $\epsilon^3 = 1$ or, for that matter, $\epsilon = 1$.

Complement: partial rotations with $\epsilon = -1$
The partial-rotation matrices $\mathcal{M}_{k_2,k_1}$ with $|k_2 - k_1| \leq \frac{p}{2}$ all share the same trivial characteristic polynomial $(t-1)^p$, but possess increasingly numerous and increasingly large Jordan blocks as $|k_2 - k_1|$ goes from 0 to $\frac{p}{2}$. For $\frac{p+1}{2} < |k_2 - k_1| < p$, the Jordan blocks disappear and the characteristic polynomials become thoroughly unremarkable, apart from being self-inverse (*always* so if p is even, *only when* $k_2 - k_1 \in \mathbb{Z}$ if p is odd). For $|k_2 - k_1| = p$, as we saw earlier in this section, we have one

[63] True, we have $V_p(1,\epsilon) = \mathit{Const} \mod 1+\epsilon+\epsilon^2$, but this is a trivial consequence of $V_p(1,\epsilon)$ being self-inverse in ϵ.

single Jordan block of maximal size, with eigenvalue ∓ 1 depending on the parity of p. That leaves only the border-line case $|k_2 - k_1| = \frac{p+1}{2}$. We have no Jordan blocks then, yet the characteristic polynomials possess a remarkable factorisation on $\mathbb{Z}$:

If $k_2 - k_1 = \pm \dfrac{p+1}{2}$ then :

$$\text{(for } p \text{ odd)} \qquad \det(t\,\mathcal{I} - \mathcal{M}_{k_2,k_1}) = (t-1) \prod_{s=1}^{\frac{p-1}{2}} P_s(p,t)$$

$$\text{(for } p = 0 \mod 4) \qquad \det(t\,\mathcal{I} - \mathcal{M}_{k_2,k_1}) = \prod_{s=1}^{\frac{p}{4}} \big(P_{2s-1}(p,t)\big)^2$$

$$\text{(for } p = 2 \mod 4) \qquad \det(t\mathcal{I} - \mathcal{M}_{k_2,k_1}) = P_{\frac{p}{2}}(p,t) \prod_{s=1}^{\frac{p-2}{4}} \big(P_{2s-1}(p,t)\big)^2$$

with polynomials $P_s(p,t) \in \mathbb{Q}[p,t]$ quadratic and self-inverse in t, of degree $2s$ in p, and assuming values in $\mathbb{Z}[t]$ for $p \in \mathbb{Z}$:

$$P_s(p,t) = \big(1-t\big)^2 + \left(\prod_{i=0}^{s-1} \frac{p-i}{1+i}\right)^2 t.$$

Complement: some properties of the polynomials H_d^δ
For any fixed $n, d \in \mathbb{N}$ with $n \le d$, the $H_d^\delta(x,n)$ and $H_d^\delta(n,y)$, as polynomials in x or y, factor into a string of fully explicitable one-degree factors. This immediately follows from the expansions (6.69), (6.70), (6.71). Conversely, the factorisations may be directly derived from the induction (6.67), (6.68) and then serve to establish the remaining properties. Most zeros (x,y) in $\mathbb{Z}^2$ or $(\frac{1}{2}\mathbb{Z})^2$ can also be read off the factorisation. All the above properties suggest a measure of symmetry between the two variables, under the simple exchange $x \leftrightarrow y$. But there also exists a more recondite symmetry, which is best expressed in terms of the polynomials

$$K_d^\delta(x,y) := H_d^\delta\left(d - x, \frac{1}{2} + 2d + \delta - \frac{3}{2}x + \frac{1}{2}y\right) \tag{6.79}$$

under the exchange $y \leftrightarrow -y$. It reads, for $x = n \in \mathbb{N} \cup [0,d]$:

$$\begin{aligned} K_d^\delta(n,y) + K_d^\delta(n,-y) &= 2^{n-1} \left[\left[\frac{d+\delta-n}{2d+\delta-2n}\right]\right]!! \prod_{0<i<n}^{i \text{ odd}} (y^2 - i^2) && (n \text{ even}) \\ &= 0 && (n \text{ odd}) \end{aligned}$$

with the factorial ratio $[[\ldots]]!!$ defined as in (6.72).

6.9 Ramified monomial inputs F: infinite order ODEs

If we now let p assume arbitrary complex values α, our *nir*-transform $h(\nu)$ and its centered variant $h_*(\nu) = h(\nu + \nu_*) = h(\nu + 1)$ ought to verify the following ODEs of infinite order

$$Q(\partial_\nu, \nu)\, h(\nu) := \left(\left(\partial_\nu - \nu\, \partial_\nu - \frac{\alpha}{2}\right)^{\alpha} - \partial_\nu^{\alpha}\right) h(\nu) = 0 \quad \alpha \in \mathbb{C} \quad (6.80)$$

$$Q_*(\partial_\nu, \nu)\, h_*(\nu) := \left(\left(-\nu\, \partial_\nu - \frac{\alpha}{2}\right)^{\alpha} - \partial_\nu^{\alpha}\right) h_*(\nu) = 0 \quad \alpha \in \mathbb{C} \quad (6.81)$$

to which a proper meaning must now be attached. This is more readily done with the first, non-centered variant, since

Proposition 6.1. *The* nir-*transform $h_\alpha(\nu)$ of $f_\alpha(x) := \alpha \log(1 + \alpha\, x)$ is of the form*

$$h_\alpha(\nu) = -h_{-\alpha}(\nu) = \frac{1}{\sqrt{2}\,\alpha} \sum_{n \in \mathbb{N}} \gamma_{-\frac{1}{2}+n}(\alpha^2)\, \nu^{-\frac{1}{2}+n} \quad (6.82)$$

with $\gamma_{-\frac{1}{2}+n}(\alpha^2)$ polynomial of degree n in α^2 and it verifies (*mark the sign change*) *an infinite integro-differential equation of the form*

$$\left(\sum_{1 \le k} \partial_\nu^{-k}\, S_k\left(\nu \partial_\nu + \frac{k}{2}, \alpha - k\right)\right) h_\alpha(-\nu) = 0 \quad (6.83)$$

with integrations ∂_ν^{-k} from $\nu = 0$ and with elementary differential operators $\mathbb{S}(.,.)$ which, being polynomial in their two arguments, merely multiply each monomial ν^n by a scalar factor polynomial in (n, k, α), effectively yielding an infinite induction for the calculation of the coefficients $\gamma_{-\frac{1}{2}+n}(\alpha^2)$.

Thus the first three coefficients are

$$\gamma_{-\frac{1}{2}}(\alpha^2) = 1,\ \ \gamma_{\frac{1}{2}}(\alpha^2) = \frac{1}{12}(\alpha^2 - 1),\ \ \gamma_{\frac{3}{2}}(\alpha^2) = \frac{1}{864}(\alpha^2 - 1)(\alpha^2 + 23).$$

For $n \ge 1$ all polynomials $\gamma_{-\frac{1}{2}+n}(\alpha^2)$ are divisible by $(\alpha^2 - 1)$ but this is their only common factor.

Remark. The regular part of the *nir*-transform h_α of f_α has the same shape $\sum_{n \in \mathbb{N}} \alpha^{-1}\, \gamma_n(\alpha^2)$ as the singular part, also with $\gamma_n(\alpha^2)$ polynomial of degree n in α^2, but it doesn't verify the integro-differential equation

(6.80). We'll need the following identies:

$$\begin{aligned}
[\mathbf{d},\mathbf{D}] &= \mathbf{d} \qquad \left(\text{here}\quad \mathbf{d}=\partial_\nu,\quad \mathbf{D}=\nu\partial_\nu+\frac{\alpha}{2}\right)\\
(\mathbf{d}+\mathbf{D})^\alpha &= \sum_{0\le k} S_k\left(\mathbf{D}+\frac{\alpha-k}{2},\alpha-k\right)\,\mathbf{d}^{\alpha-k}\\
&= \sum_{0\le k} \mathbf{d}^{\frac{\alpha-k}{2}}\; S_k(\mathbf{D},\alpha-k)\;\mathbf{d}^{\frac{\alpha-k}{2}}\\
&= \sum_{0\le k} \mathbf{d}^{\alpha-k}\; S_k\left(\mathbf{D}-\frac{\alpha-k}{2},\alpha-k\right).
\end{aligned} \tag{6.84}$$

The non-commutativity relation $[\mathbf{d},\mathbf{D}]=1$, combined with the above expansions, yields for the polynomials S_k the following addition equation:

$$S_k(\mathbf{D},\beta_1+\beta_2)=\sum_{k_1+k_2=k} S_{k_1}\left(\mathbf{D}-\frac{\beta_2-k_2}{2},\beta_1-k_1\right)S_{k_2}\left(\mathbf{D}+\frac{\beta_1-k_1}{2},\beta_2-k_2\right)$$

and the difference equation:

$$S_k\left(\frac{\beta}{2},\beta\right)\equiv S_k\left(\frac{\beta+1}{2},\beta-1\right). \tag{6.85}$$

That relation, in turn, has two consequences: on the one hand, it leads to a finite expansion (6.86) of $S_k(\mathbf{D},\beta)$ in powers of $\mathbf{D}$ with coefficients $T_{2k_*}(\beta)$ that are polynomials in β of degree exactly k_* with $2k_*\le k$. On the other, it can be partially reversed, leading, for entire values of b, to a finite expansion (6.87) of $T_{2k}(b)$ in terms of some special values of $S_{2k-1}(.\,,b)$.

$$S_k(\mathbf{D},\beta)=\left(\prod_{i=1}^{k}(\beta+i)\right)\sum_{k_1+2k_2=k}^{k_1,k_2\ge0}\frac{\mathbf{D}^{k_1}}{k_1!}\frac{T_{2k_2}(\beta)}{(2k_2)!}\quad\forall\beta\in\mathbb{C} \tag{6.86}$$

$$T_{2k}(b)=\frac{(2k)!\,b!}{(2k+b)!}\sum_{0\le c\le b}\left(c-\frac{b}{2}\right)S_{2k-1}\left(\frac{c-b}{2},c\right)\quad\forall b\in\mathbb{N}. \tag{6.87}$$

Together, (6.86) and its reverse (6.87) yield an explicit inductive scheme for constructing the polynomials T_{2k}. We first calculate $T_{2k}(b)$ for b whole, via the identity (6.88) whose terms $S_{2k-1}(.\,,b)$ involve only the earlier polynomials $T_{2h}(c)$, with indices $h<k$ and $c\le b$. The identity reads:

$$\frac{T_{2k}(b)}{(2k)!b!}=\sum_{0\le c\le b}^{0\le h<k}\frac{T_{2h}(c)}{(2h)!c!}\,\frac{(c/2-b/2)^{2k-2h-1}}{(2k-2h-1)!}\,\frac{(2k+c)!}{(2k+b)!}\,\frac{(c-b/2)}{(c+2k)}. \tag{6.88}$$

Then we use Lagrange interpolation (6.89)-(6.90) to calculate $T_{2k}(\beta)$ for general complex arguments β :

$$T_{2k}(\beta) = \sum_{1\le b\le k} \Lambda_k(\beta,b)\; T_{2k}(b) \qquad \forall\beta\in\mathbb{C} \tag{6.89}$$

$$\Lambda_k(\beta,b) := \frac{\beta}{b}\prod_{1\le i\le k}^{i\neq b}\frac{i-\beta}{i-b} \tag{6.90}$$

First values of the $T_{2\,k}$-polynomials

$$T_0(\beta) = 1$$
$$T_2(\beta) = \frac{1}{12}\,\beta$$
$$T_4(\beta) = \frac{1}{240}\,\beta\,(-2+5\,\beta)$$
$$T_6(\beta) = \frac{1}{4032}\,\beta\,(16+42\,\beta+35\,\beta^2)$$
$$T_8(\beta) = \frac{1}{34560}\,\beta\,(-4+5\,\beta)(36-56\,\beta+35\,\beta^2)$$
$$T_{10}(\beta) = \frac{1}{101376}\,\beta\,(768-2288\,\beta+2684\,\beta^2-1540\,\beta^3+385\,\beta^4).$$

Special values of the $T_{2\,k}$-polynomials

$$T_{2\,k}(2) = \frac{2}{(2\,k+1)(2\,k+2)}$$
$$T_{2\,k}(1) = \frac{1}{(2\,k+1)\,2^{2\,k}}$$
$$T_{2\,k}(0) = 0 \quad \text{if} \quad k\neq 0 \quad \text{and} \quad T'_{2\,k}(0) = \frac{B_{2\,k}}{2\,k}$$
$$T_{2\,k}(-1) = B_{2\,k}(\frac{1}{2})$$
$$T_{2\,k}(-2) = -(2\,k-1)\;B_{2\,k}$$
$$T_{2\,k}(-1-2\,k) = (-1)^k\,\frac{(2\,k)!}{4^k\,k!}$$

with B_n and $B_n(.)$ denoting the Bernoulli numbers and polynomials.

Special values of the S_k-polynomials

$$\text{For } k \text{ odd}: \quad S_k(\mathbf{D}, -1-k) = \prod_{-\frac{k}{2}<s<\frac{k}{2}}^{k\in\mathbb{Z}} (\mathbf{D}+s) \tag{6.91}$$

$$\text{For } k \text{ even}: \quad S_k(\mathbf{D}, -1-k) = \prod_{-\frac{k}{2}<s<\frac{k}{2}}^{k\in\mathbb{Z}-\frac{1}{2}\mathbb{Z}} (\mathbf{D}+s). \tag{6.92}$$

Note that in neither case are the bounds $\pm k/2$ reached by s, since the pair $\{k/2, s\}$ always consists of an integer and a half-integer. $S_k(\mathbf{D}, b)$ appears to have no simple factorisation structure except (trivially) for $b = 1$ and $b = 2$ when in view of (6.86),(6.87) we have:

$$\begin{aligned} S_k(\mathbf{D}, 1) &= 2^{-k-1}\Big((2\,\mathbf{D}+1)^{k+1} - (2\,\mathbf{D}-1)^{k+1}\Big) \\ S_k(\mathbf{D}, 2) &= 2^{-1}\Big((\mathbf{D}+1)^{k+2} + (\mathbf{D}-1)^{k+2} - 2\,\mathbf{D}^{k+2}\Big). \end{aligned}$$

Since $S_1(n, \alpha-1) = n\,\alpha$, the induction rule for the γ-coefficients reads

$$\begin{aligned} &\gamma_{-\frac{1}{2}}(\alpha^2) = 1 \\ &\gamma_{-\frac{1}{2}+n}(\alpha^2) = \sum_{1\le k\le n} (-1)^{k+1} \frac{\Gamma\left(\frac{1}{2}+n-k\right)}{\Gamma\left(\frac{1}{2}+n\right)} \frac{S_{k+1}\left(n-\frac{k}{2}, a-k-1\right)}{S_1(n, \alpha-1)} \gamma_{-\frac{1}{2}+n-k}(\alpha^2) \\ &= \sum_{1\le k\le n} (-1)^{k+1} \frac{\Gamma\left(\frac{1}{2}+n-k\right)}{\Gamma\left(\frac{1}{2}+n\right)} \frac{S_{k+1}\left(n-\frac{k}{2}, a-k-1\right)}{n\,\alpha} \gamma_{-\frac{1}{2}+n-k}(\alpha^2). \end{aligned}$$

Moreover, since $S(k)(\mathbf{D}, -1) = S(k)(\mathbf{D}, -2) = \cdots = S(k)(\mathbf{D}, -k) = 0$, when α is a positive integer, the above induction involves a constant, finite number of terms, with a sum $\sum$ over $1 \le k \le \alpha - 1$ instead of $1 \le k \le n$, which is consistent which the finite differential equations of Section 6.6.

6.10 Ramified monomial inputs F: arithmetical aspects

In this last subsection, we revert to the case of polynomial inputs f and replace the high-order ODEs verified by k by a first-order order differential system, so as to pave the way for a future paper [16] devoted to understanding, from a pure ODE point of view, the reasons for the rigidity of the *inner algebra's* resurgence, *i.e.* its surprising insentivity to the numerous parameters inside f.

The normalised coefficients γ_r, δ_r, δ_r^{ev} of the series h_α, k_α, k_α^{ev}, whose definitions we recall:

$$h_\alpha(\nu) = \frac{1}{\sqrt{2}}\frac{1}{\alpha}\sum_{r\in -\frac{1}{2}+\mathbb{N}} \gamma_r(\alpha^2)\;\nu^r \qquad \text{with } \gamma_{-\frac{1}{2}}(\alpha^2)\equiv 1$$

$$k_\alpha(n) = \frac{\sqrt{\pi}}{\sqrt{2}}\frac{1}{\alpha}\sum_{r\in -\frac{1}{2}+\mathbb{N}} \delta_r(\alpha^2)\;n^{-r} \qquad \text{with } \delta_{-\frac{1}{2}}(\alpha^2)\equiv 1$$

$$k_\alpha(n)k_\alpha(-n) =: k_\alpha^{\mathrm{ev}}(n) = \frac{\pi i}{2\alpha^2}\sum_{r\in -1+2\,\mathbb{N}} \delta_r^{\mathrm{ev}}(\alpha^2)\;n^{-r} \qquad \text{with } \delta_{-1}^{\mathrm{ev}}(\alpha^2)\equiv 1$$

seem to possess remarkable arithmetical properties, whether we view them

(i) as polynomials in α;
(ii) as polynomials in r;
(iii) as rational numbers, for α fixed in $\mathbb{Z}$.

These arithmetical properties, at least some of them, do not obviously follow from the shape of the *nir* transform nor indeed from the above induction. Thus, as *polynomials*, the γ coefficients appear to be exactly of the form:

$$\gamma_{-\frac{1}{2}+r}(\alpha^2) = \frac{6^{-r}}{(2r)!}\frac{(\alpha^2-1)\gamma^*_{-\frac{1}{2}+r}(\alpha^2)}{\prod\limits_{5\le p\le r+2} p^{\mu_{r,p}}} = \frac{1}{(2r)!}\frac{a^r}{6^r}\gamma^{**}_{-\frac{1}{2}+r}(\alpha^2-1) \quad (6.93)$$

with the $\gamma^*_{-\frac{1}{2}+r}(\alpha^2)$ irreducible in $\mathbb{Z}[\alpha^2]$ and with on the denominator a product $\prod$ involving only prime numbers between 5 and $r+2$. This at any rate holds for all values of n up to 130. The surprising thing is not the presence of these p in $[5, r+2]$ but rather the fact that their powers $\mu_{r,p}$ seem to obey no exact laws (though they are easily majorised), unlike the powers of 2 and 3 that are *exactly* accounted for by the factor 6^{-r}. But this 2- and 3-adic regularity seems to go much further. It becomes especially striking if we consider the polynomials $\gamma^{**}_{-\frac{1}{2}+r}$ in the rightmost term of (6.93) after changing to the variable $a := \alpha^2 - 1$. Indeed:

Conjecture 6.1 (2- or -adic expansions for the γ as α-polynomials). The polynomials γ^{**} defined by

$$\gamma_{-\frac{1}{2}+r}(a+1) =:, \frac{1}{(2r)!}\frac{a^r}{6^r}\gamma^{**}_{-\frac{1}{2}+r}(a), \qquad \gamma^{**}_{-\frac{1}{2}+r} \in \mathbb{Q}[a^{-1}] \qquad (6.94)$$

possess 2- and 3-adic expansions to all orders:

$$\gamma^{**}_{-\frac{1}{2}+r}(a) = \sum_{0\le j} \lambda_{2,j}(r,a)\, 2^j \qquad \Big(\lambda_{2,j}(r,a) \in \{0,1\}\Big)$$
$$\gamma^{**}_{-\frac{1}{2}+r}(a) = \sum_{0\le j} \lambda_{3,j}(r,a)\, 3^j \qquad \Big(\lambda_{3,j}(r,a) \in \{0,1,2\}\Big)$$

with coefficients $\lambda_{p,j}$ that in turn depend only on the first j terms of the p-adic expansion of r. In other words:

$$\lambda_{2,j}(r,a) = \lambda_{2,j}([r_0, r_1, \ldots, r_{j-1}], a) \quad \text{with} \quad r = \sum_{0\le i<j} r_i\, 2^i \quad \mod 2^j$$
$$\lambda_{3,j}(r,a) = \lambda_{3,j}([r_0, r_1, \ldots, r_{j-1}], a) \quad \text{with} \quad r = \sum_{0\le i<j} r_i\, 3^i \quad \mod 3^j.$$

Moreover, as a polynomial in a^{-1}, each $\lambda_{2,j}(r,a)$ is of degree j at most.

These facts have been checked up to the p-adic order $j = 25$ and for all r up to 130. Moreover, no such regularity seems to obtain for the other p-adic expansions, at any rate not for $p = 5, 7, 11, 13$.

Conjecture 6.2 (p-adic expansions for the γ as r-polynomials). The γ^{**} defined as above verify

$$\gamma^{**}_{-\frac{1}{2}+r}(a) = 1 + \sum_{1\le d\le r} a^{-d}\, Q_d(r) \tag{6.95}$$

with universal polunomials $Q_d(r)$ of degree $3d$ in r and of the exact arithmetical form:

$$Q_d(r) = \Big(6^d \prod_{p\,\text{prime}\ge 2} p^{-\mu_p(d)}\Big) \Big(Q_d^*(r) \prod_{1\le i\le d} (r-i)\Big)$$
$$Q_d^*(r) = \sum_{1\le i\le 2d} c_{d,i}\, r^i \qquad \text{with} \qquad (c_{d,1}, \ldots, c_{d,2d}) \quad \text{coprime}$$
$$\mu_p(d) = \sum_{0\le s} \text{en}\left(\frac{2\,d}{(p-3)\,p^s}\right) \qquad \text{if} \quad p \ge 5$$
$$\mu_3(d) = \sum_{0\le s} \text{en}\left(\frac{d}{2\,p^s}\right)$$
$$\mu_2(d) = -\sum_{0\le s} d_s \qquad \text{if} \quad d = \sum_{0\le s} d_s\, 2^s \quad (d_s \in \{0,1\})$$
$$c_{d,2d} \in (-1)^d \mathbb{N}^+.$$

With $\text{en}(x)$ denoting as usual the entire part of x.

Conjecture 6.3 (Special values of the Q_d). If for any $q \in \mathbb{Q}$ we set

$$\begin{aligned} &\mathrm{pri}(q) := q \quad \text{if} \quad q \in \mathbb{N} \quad \text{and} \quad q \quad \text{prime} \\ &\mathrm{pri}(q) := 1 \quad \text{otherwise} \end{aligned}$$

then for any $d, s \in \mathbb{N}^*$ we have

$$Q_d(d+s) \in \frac{1}{Q_{d,s}}\mathbb{Z} \quad \text{with} \quad Q_{d,s} := \prod_{\substack{1 \le j \le s \\ s<k \le s+2\,j}} \mathrm{pri}\left(\frac{d+k}{j}\right). \tag{6.96}$$

Moreover, for s fixed and d large enough, the denominator of $Q_d(d+s)$ is exactly $Q_{d,s}$. Note that by construction $Q_d(d+s)$ is automatically quadratfrei as soon as $d > 2\,s\,(s-1)$.

Together with the trivial identities $Q_d(s) = 0$ for $s \in [0, d] \cup \mathbb{N}$, this majorises *denom*$(Q_d(s)$ for all $s \in \mathbb{N}$. We have no such simple estimates for negative values of s.

Conjecture 6.4 (Coefficients δ^{ev}). The normalised coefficients δ_r of k_α (with $r \in -1 + 2\,\mathbb{N}$) are of the form:

$$\delta_r^{\mathrm{ev}}(\alpha^2) = \frac{1}{B_r}\, R_r(\alpha^2) = \frac{A_r}{B_r}\, R_r^*(\alpha^2) \prod_{d|r}(\alpha^2 - d^2) \tag{6.97}$$

where

(i) A_r is of the form $\prod_{p|r}^{p\ \mathrm{prime}} p^{\sigma_{p,r}}$ with $\sigma_{r,p} \in \mathbb{N}$;
(ii) B_r is of the form $\prod_{(p,r)=1}^{p\ \mathrm{prime} \le r+2} p^{\tau_{p,r}}$ with $\tau_{r,p} \in \mathbb{N}$;
(iii) $R_r^*(\alpha^2)$ is an irreducible polynomial in $\mathbb{Z}[\alpha^2]$.
However, when α takes entire values q, the arithmetical properties of $\delta_r^{\mathrm{ev}}(q^2)$ become more dependent on q than r. In particular;
(iv) $\mathrm{denom}(\delta_r(q^2)) = \prod_{p|q}^{p\ \mathrm{prime}} d^{\kappa_{r,q,p}}$ with $\kappa_{r,q,p} \in \mathbb{N}$ and $\kappa_{r,p,p} \le 3$;
(v) $\mathrm{denom}(\delta_r(p^2)) = p^{\kappa_{r,p,p}}$ with $\kappa_{r,p,p} \le 3$ (p prime). This suggests a high degree of divisibility for $R_r^*(q^2)$ and above all $R_r(q^2)$, specially for q prime. In particular we surmise that:
(vi) $R_r(p^2) \in (p-1)!\, \mathbb{Z}$ for p prime.

6.11 From flexible to rigid resurgence

Let us, in this concluding subsection, revert to the case of polynomial inputs f. We assume the tangency order to be 1. To get rid of the demi-entire powers, we go from k to a new unknown K such that

$$k(n) = n^{\frac{1}{2}}\ K(n) \qquad (k(n) \in n^{\frac{1}{2}}\mathbb{C}[[n^{-1}]]\ ,\ K(n) \in \mathbb{C}[[n^{-1}]]). \tag{6.98}$$

The new differential equation in the n-plane reads $P(n, -\partial_n)K(n) = 0$ and may be written in the form:

$$\begin{aligned}
&\left(\prod_{i=1}^{r}(\partial_n + \nu_i)\right) K(n) + \sum_{i=0}^{r-1} \theta_i^*(n)\, \partial_n^i K(n) = 0;\\
&\theta_i^*(n) = O(n^{-1}).
\end{aligned} \tag{6.99}$$

This ODE is equivalent to the following first order differential system with r unknowns $K_i^* = \partial_n^i K$ $(0 \le i \le r-1)$:

$$\begin{aligned}
\partial_n K_0^* - K_1^* &= 0\\
\partial_n K_1^* - K_2^* &= 0\\
&\dots\\
\partial_n K_{r-2}^* - K_{r-1}^* &= 0\\
\partial_n K_{r-1}^* + \sum_{i=0}^{r-1} \left(\nu_{r-i}^* + \theta_i^*(n)\right) K_i &= 0
\end{aligned} \tag{6.100}$$

with ν_l standing for the symmetric sum of order l of $\nu_1, \dots, \nu_r$. Changing from the unknowns K_i^* $(0 \le i \le r-1)$ to the unknowns K_i $(1 \le i \le r)$ under the vandermonde transformation

$$K_i^* = \sum_{0 \le j \le r-1} (-\nu_i)^j\, K_j = \sum_{0 \le j \le r-1} (-\nu_i)^j\, \partial_n^j K$$

we arrive at a new differential system in normal form:

$$\partial_n K_i + \nu_i\, K_i + \sum_{i=1}^{r} \theta_{i,j}(n)\, K_j = 0 \qquad (1 \le i \le r) \tag{6.101}$$

with constants ν_i and rational coefficients $\tau_i(n)$ which, unlike the earlier $\theta_i(n)$, are not merely $O(n^{-1})$ but also, crucially, $O(n^{-2})$. Concretely, the rank-1 matrix $\Theta = [\theta_{i,j}]$ is conjugate to a rank-1 matrix $\Theta^* = [\theta_{i,j}^*]$ with only one non-vanishing (bottom) line, under the vandermonde matrix $V = [v_{i,j}]$:

$$\begin{aligned}
&\Theta = V^{-1}\, \Theta^*\, V \quad \text{with} \quad v_{i,j} = (-\nu_i)^{j-1},\\
&\theta_{i,j}^* = 0 \;\text{ if } i < r,\; \theta_{r,j}^* = \theta_{j-1}^*.
\end{aligned} \tag{6.102}$$

The coefficients $\theta_{i,j}$ have a remarkable structure. They admit a unique factorisation of the form:

$$\theta_{i,j}(n) = \frac{1}{\delta_i}\, \frac{\alpha_j(n)}{\gamma(n)} \tag{6.103}$$

with factors α_i, β_j, γ derived from symmetric polynomials α, β, γ of r–1 f-related variables (not counting the additional n-variable):

$$\begin{aligned}
\delta_i \quad &= \delta(x_1 - x_i, \ldots, \widehat{x_i - x_i}, \ldots, x_r - x_i) && (6.104)\\
\alpha_j(n) &= \alpha(x_1 - x_j, \ldots, \widehat{x_j - x_j}, \ldots, x_r - x_j)(n) && (6.105)\\
\gamma(n) &= \gamma(x_1 - x_i, \ldots, \widehat{x_i - x_i}, \ldots, x_r - x_i)(n) && (6.106)\\
&= \gamma(x_1 - x_j, \ldots, \widehat{x_j - x_j}, \ldots, x_r - x_j)(n). && (6.107)
\end{aligned}$$

Let us take a closer look at all three factors:

(i) the γ factor is simply the "second leading polynomial" of Section 6.2 and Section 6.3 after division by its leading term $n^{\overline{\delta}}$. Being a direct shift-invariant of f, if may also be viewed as a polynomial in $\mathbf{f}_0, \mathbf{f}_1 \ldots$;

(ii) the δ factor comes from the inverse vandermonde matrix $V^{-1} = [u_{i,j}]$. Indeed:

$$u_{i,j} = \sigma_{r-j}(\nu_1, \ldots, \nu_r)\, \delta_i \quad \text{with} \quad \delta_i = \prod_{1\le s\le r}^{s\neq i} \frac{1}{\nu_s - \nu_i} = u_{i,r}. \qquad (6.108)$$

Moreover:

$$\begin{aligned}
\nu_i &= f^*(x_i) = f_r \int_0^{x_i} \prod_{1\le j\le r}^{j\neq i} (x - x_j)\, dx\\
\nu_i - \nu_j &= \nu(x_i, x_j\,;\, x_1 \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_r)
\end{aligned}$$

with a function ν antisymmetric (respectively symmetric) in its first two (respectively last $r-2$) variables, and completely determined by the following two identities:

$$\nu(x_1, x_2; y_1, \ldots, y_{r-2}) \equiv \nu(x_1 - t, x_2 - t; y_1 - t, \ldots, y_{r-2} - t) \quad \forall t$$

$$\nu(x, -x;\; y_1, \ldots, y_{r-2}) = -\nu(-x, x;\; y_1, \ldots, y_{r-2})$$

$$= (-1)^{r-1}\, 4\, f_r \sum_{1\le s\le [\frac{r-2}{2}]} \frac{\sigma_{r-2s}(y_1, \ldots, y_{r-2})}{4\, s^2 - 1} t^{2s+1}$$

$$= -4 f_r \int_0^x t_2\, dt_2 \int_0^{t_2} \left(\prod_{i=1}^{r-2} (y_i + t_1) + \prod_{i=1}^{r-2} (y_i - t_1) \right) dt_1$$

where $\sigma_l(y_1, y_2, \dots)$ denotes the symmetric sum of order l of y_1, $y_2 \dots$;

(iii) the α factor stems from the coefficients θ_s in the differential equation (6.101). Indeed, in view of (i) and (ii) and with the n-variable implicit:

$$\theta_{i,j} = \frac{1}{\delta_i}\frac{\alpha_j(n)}{\gamma(n)} = \sum_{\substack{1\le t\le r\\1\le s\le r}} u_{i,t}\theta^*_{t,s} v_{s,j} = \sum_{1\le s\le r} u_{i,r}\theta^*_{r,s}\, v_{s,j}$$

$$= \sum_{1\le s\le r} \frac{1}{\delta_i}\,\theta^*_{s-1}\,(-\nu_j)^{s-1}.$$

Hence

$$\alpha_j(n) = \gamma(n) \sum_{0\le l\le r-1} (-\nu_j)^l\, \theta^*_l(n). \tag{6.109}$$

We may also insert the covariant shift ν_0 or $\underline{\nu}_0$ and rewrite the differential equation (6.101) as:

$$\left(\prod_{i=1}^{r}(\partial_n + \nu_i)\right) K(n) + \sum_{i=0}^{r-1} \theta^\#_i(n)\,(\partial_n + \nu_0)^i K(n) = 0 \tag{6.110}$$

$$\left(\prod_{i=1}^{r}(\partial_n + \nu_i)\right) K(n) + \sum_{i=0}^{r-1} \underline{\theta}^\#_i(n)\,(\partial_n + \underline{\nu}_0)^i K(n) = 0 \tag{6.111}$$

with new coefficients $\theta^\#_i$, $\underline{\theta}^\#_i$ that are not only shift-invariant but also root-symmetric.[64] This leads for the α factors to expressions:

$$\alpha_j(n) = \gamma(n) \sum_{0\le l\le r-1} (\nu_0 - \nu_j)^l\, \theta^\#_l(n) = \gamma(n) \sum_{0\le l\le r-1} (\underline{\nu}_0 - \nu_j)^l\, \underline{\theta}^\#_l(n)$$

which have over (6.105) the advantage of involving only shift-invariants, namely the $\theta^\#_l$ or $\underline{\theta}^\#_j$ (root-symmetric) and the $\nu_0 - \nu_j$ or $\underline{\nu}_0 - \nu_j$ (not root-symmetric). To show the whole extent of the *rigidity*, we may even introduce new parameters by taking a non-standard shift operator $\beta(\partial_\tau)$,

$$\beta(t) = t^{-1} + \sum_{0\le k} \beta_k\, t^k = t^{-1} + \sum_{1\le k} b_k\, t^{k-1} \qquad (b_k \equiv \beta_{k-1})$$

[64] *I.e.* symmetric with respect to the roots x_i of f.

but with a re-indexation $b_k = \beta_k$ to do justice to the underlying homogeneity.[65] The case $r = 1$ is uninteresting (no ping-pong, there being only one inner generator), and here are the results for $r = 2$ and 3.

Input f of degree 2: $\quad f(x) = (x - x_1)\,(x - x_2)\,f_2$

$$\delta(y_1) = -\frac{1}{6}\, f_2\, y_1^3$$

$$\gamma(y_1) = +36\, f_2^4\, y_1^2 + 288\, f_2^4\, b_2\, n^{-2}$$

$$\begin{aligned}\alpha(y_1) = &+(5\, f_2^4\, y_1^2 - 3\, f_2^6\, y_1^6\, b_2)\, n^{-2} + 8\,(f_2^6\, y_1^5\, b_3 - 12\, f_2^5\, b_2\, y_1^3)\, n^{-3}\\ &+ 8\,(9\, f_2^5\, y_1^2\, b_3 - 7\, f_2^4\, b_2 - 3\, f_2^6\, y_1^4\, b_2^2)\, n^{-4}\\ &+ 64\, f_2^6\, y_1^3\, b_2\, b_3\, n^{-5} + 24\,\big(12\, f_2^5\, b_2\, b_3 + f_2^6\, y_1^2\,(4\, b_2^3 + b_3^2)\big)\, n^{-6}\\ &+ 128\, f_2^6\, b_2\,(4\, b_2^3 + b_3^2)\, n^{-8}.\end{aligned}$$

Input f of degree 3: $\quad f(x) = (x - x_1)\,(x - x_2)\,(x - x_3)\,f_3$
$\mathbf{y}_1 := y_1 + y_2\,,\ \mathbf{y}_2 := y_1\, y_2$

$$\delta(y_1, y_2) = \frac{1}{144}\, f_3^2\, \mathbf{y}_2^3\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)$$

$$\begin{aligned}\gamma(y_1, y_2) = &-2^6\, 3^3\, f_3^{10}\, \mathbf{y}_1\, \mathbf{y}_2^2\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(4\,\mathbf{y}_2 - \mathbf{y}_1^2)\\ &- 2^6\, 3^3\, f_3^9\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)\, n^{-1}\\ &+ 2^9\, 3^3\, f_3^{10}\, b_2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)^2\, n^{-2}\\ &+ 2^6\, 3^3\,\big(243\, f_3^9\, \mathbf{y}_1\, b_2\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\\ &+ f_3^{10}\, b_3\,(513\,\mathbf{y}_1^2\,\mathbf{y}_2^2 + 28\,\mathbf{y}_1^6 - 252\,\mathbf{y}_1^4\,\mathbf{y}_2 + 216\,\mathbf{y}_2^3)\big)\, n^{-3}\\ &- 2^6\, 3^6\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)\,\big(33\, f_3^9\, b_3 + 4\, f_3^{10}\, b_2^2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\big)\, n^{-4}\\ &- 2^8\, 3^7\, f_3^{10}\, b_2\, b_3\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)^2\, n^{-5}\\ &+ 2^6\, 3^9\,\big(9\, f_3^9\, b_2\, b_3 - f_3^{10}\, b_3^2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\big)\, n^{-6}\\ &+ 2^6\, 3^{12}\, f_3^{10}\, b_3\,(4\, b_2^3 + b_3^2)\, n^{-9}.\end{aligned}$$

[65] N.B. the present b_k differ from those in (6.43).

For the α factor, we mention only the two lowest and highest powers of n^{-1}:

$$\begin{aligned}\alpha(y_1,y_2) = \Bigg(&\frac{1}{27} f_3^{11}\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(8748\,\mathbf{y}_2^5 - 13851\,\mathbf{y}_1^2\,\mathbf{y}_2^4 + 378\,\mathbf{y}_1^4\,\mathbf{y}_2^3\\
&+2403\,\mathbf{y}_1^6\,\mathbf{y}_2^2 - 600\,\mathbf{y}_1^8\,\mathbf{y}_2 + 40\,\mathbf{y}_1^{10}) + \frac{2}{27} f_3^{13}\, b_2\,\mathbf{y}_1\,\mathbf{y}_2^2\\
&\times (9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)(4\,\mathbf{y}_2 - \mathbf{y}_1^2)(9\,\mathbf{y}_2^2 + 6\,\mathbf{y}_1^2\,\mathbf{y}_2 - \mathbf{y}_1^4)\\
&\times (81\,\mathbf{y}_2^3 - 36\,\mathbf{y}_1^2\,\mathbf{y}_2^2 + 9\,\mathbf{y}_1^4\,\mathbf{y}_2 - \mathbf{y}_1^6)\Bigg)\, n^{-2}\\
&+\Bigg(f_3^{10}\,\mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(2835\,\mathbf{y}_2^3 - 675\,\mathbf{y}_1^2\,\mathbf{y}_2^2 - 9\,\mathbf{y}_1^4\,\mathbf{y}_2 + \mathbf{y}_1^6)\\
&+\frac{2}{27} f_3^{12}\, b_2\,\mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(66339\,\mathbf{y}_2^6 - 129033\,\mathbf{y}_1^2\,\mathbf{y}_2^5\\
&+175770\,\mathbf{y}_1^4\,\mathbf{y}_2^4 - 119475\,\mathbf{y}_1^6\,\mathbf{y}_2^3 + 38520\,\mathbf{y}_1^8\,\mathbf{y}_2^2 - 5742\,\mathbf{y}_1^{10}\,\mathbf{y}_2\\
&+319\,\mathbf{y}_1^{12}) - \frac{3}{9} f_3^{13}\, b_3\,\mathbf{y}_1^2\,\mathbf{y}_2^2\,(4\,\mathbf{y}_2 - \mathbf{y}_1^2)\,(9\,\mathbf{y}_2^3 - 18\,\mathbf{y}_1^2\,\mathbf{y}_2^2\\
&+9\,\mathbf{y}_1^4\,\mathbf{y}_2 - \mathbf{y}_1^6)(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)^2\Bigg)\, n^{-3} + \sum_{4\le s\le 18} (\ldots)\, n^{-s}\\
&+2\times 3^{14}\, f_3^{13}\, b_3\,(4\,b_2^3 + b_3^2)\,(b_2\,b_4 - 12\,b_2^3 - 3\,b_3^2)\,b_4\\
&\times (3\,\mathbf{y}_2 - \mathbf{y}_1^2)\, n^{-19} + \frac{3^{15}}{2} f_3^{13}\, b_3\,(4\,b_2^3 + b_3^2)\\
&\times \big(27\,(4\,b_2^3 + b_3^2 - b_2\,b_4)^2 - b_4^3\big)\, n^{-21}.\end{aligned}$$

For a direct, ODE-theoretical derivation of the *rigidity* phenomenon, see [16]. General criteria will also be given there for deciding which parameters inside an ODE contribute to the resurgence constants (or Stokes constants) and which don't.

7 The general resurgence algebra for SP series

We recall the definition of the *raw* and *cleansed* SP series:

$$j_F(\zeta) := \sum_{0\le n} J_F(n)\,\zeta^n \quad \text{with} \quad J_F(n) := \sum_{0\le m<n}\ \prod_{0\le k\le m} F\left(\frac{k}{n}\right) \tag{7.1}$$

$$j_F^{\#}(\zeta) := \sum_{0\le n} J_F^{\#}(n)\,\zeta^n \quad \text{with} \quad J_F^{\#}(n) := J_F(n)/Ig_{_F}(n). \tag{7.2}$$

We also recall that the $\perp$ transform turns the set $\{F, f, f^*, j_F^\#\}$ into the set $\{F^\vDash, f^\vDash, f^{\vDash *}, j_{F^\vDash}^\#\}$ with:

$$F^\vDash(x)=1/F(1-x); f^\vDash(x)=-f(1-x); f^{\vDash *}(x)=f^*(1-x)-\eta_F \quad (7.3)$$

$$j_{F^\vDash}^\#(\zeta)=j_F^\#\left(\frac{\zeta}{\omega_F}\right) \quad \text{with } \omega_F := e^{-\eta_F} \text{ and } \eta_F := \int_0^1 f(x)dx. \quad (7.4)$$

and with f^* denoting as usual the primitive of f that vanishes at 0. We shall now (pending a more detailed investigation in [15]) sketch how the various generators arise and how they reproduce under alien differentiation. Piecing all this information together, we shall then get a global description of the Riemann surfaces of our SP functions. For convenience, let us distinguish two degrees of difficulty:

- first, the case of holomorphic inputs f;
- second, the case of meromorphic inputs F;

and split the investigation into two phases:

- first, focusing on the auxiliary ν-plane;
- second, reverting to the original ζ-plane.

7.1 Holomorphic input f. The five arrows

7.1.1 From *original* to *outer* Let us check, in the four simplest instances, that SP series (our so-called *original* generators) with an holomorphic input f always give rise to two *outer* generators[66]

$$\{\overset{\wedge}{lo}_{\text{in}}(\nu), \overset{\wedge}{Lo}_{\text{in}}(\zeta) = \overset{\wedge}{lo}_{\text{in}}(1+\zeta)\}, \quad \{\overset{\wedge}{lo}_{\text{out}}(\nu), \overset{\wedge}{Lo}_{\text{out}}(\zeta) = \overset{\wedge}{lo}_{\text{out}}(1+\omega_F\zeta)\}$$

located respectively over $\{\nu = 0, \zeta = 1\}$ or $\{\nu = \eta_F, \zeta = 1/\omega_F\}$ and produced under the *nur*-tranform, *i.e.* by inputting respectively f or $f^\vDash$ into the long chain of Section 5.2

Case 7.1 ($-f^*$ decreases on $[0, 1]$**).** To explain the occurence Lo_{in}, apply the argument at the beginning of Section 5.1. To explain the occurence Lo_{out}, the shortest way is to pick $\epsilon > 0$ small enough for $-f^*$ to be decreasing on the whole of $[0, 1+\epsilon]$, and then to form the SP series $jj_F^\#(\zeta)$ defined exactly as $j_F^\#(\zeta)$ but with a summation ranging over $0 \le m < (1+\epsilon)n$ instead of $0 \le m < n$. Then $jj_F^\#$ clearly has no

[66] Which exceptionally coalesce into one when $\eta_F = 0, \omega_F = 1$, which may occur only in the cases 3 or 4 *infra*.

singularity at $\zeta = 1/\omega_F$. On the other hand, applying once again the argument of Section 5.1 to the difference $jj_F^\# - j_F^\#$ we see that it has at $\zeta = 1/\omega_F$ a singularity which, up to the dilation factor ω_F, is given by the *nur*-transform of ${}^1 f$ with ${}^1 f(x) := f(1+x)$. In view of the parity relation of Section 5.8 it is also equal to *minus* the *nur*-transform of $({}^1 f)^{\vdash}$. But $({}^1 f)^{\vdash} = f^{\vDash}$. Hence the result.[67]

A trivial - but telling - example corresponds to the choice of a constant input $F(x) \equiv \alpha$ with $0 < \alpha < 1$. We then get:

$$\begin{aligned} j_F(\zeta) &= \frac{\alpha}{1-\alpha}\left(\frac{1}{1-\zeta} - \frac{1}{1-\alpha\zeta}\right); \\ j_F^\#(\zeta) &= \frac{1}{\alpha^{-1/2} - \alpha^{1/2}}\left(\frac{1}{1-\zeta} - \frac{1}{1-\alpha\zeta}\right). \end{aligned} \tag{7.5}$$

Case 7.2 ($-f^*$ increases on $[0, 1]$**).** The $\vDash$ transform turns Case 7.2 into Case 7.1, with f anf $f^{\vDash}$ exchanged. Hence the result. Again, we have the trivial example of a constant input $F(x) \equiv \alpha$ but now with $1 < \alpha$. We then get the same power series as in (7.5) but with α changed into $1/\alpha$, which of course agrees with the relation (7.4) between $j_F^\#$ and $j_{F^{\vDash}}^\#$.

Case 7.3 ($-f^*$ decreases on $[0, x_0]$**, then increases on** $[x_0, 1]$**).** Here again, the argument at the beginning of Section 5.1 takes care of Lo_{in}. To justify Lo_{out}, all we have to do is observe that the transform $\vDash$ turns Case 7.3 into another instance of that same Case 7.3, while exchanging the roles of Lo_{in} and Lo_{out}.

Case 7.4 ($-f^*$ increases on $[0, x_0]$**, then decreases on** $[x_0, 1]$**).** Case 7.4 is exactly the reverse of Case 7.3. The argument about $j_F^\#$ and $jj_F^\#$ (see Case 7.1) takes care of Lo_{out} and then the fact that $\vDash$ turns Case 7.4 into another Case 7.4 justifies the occurence of Lo_{in}. Case 7.4, however, presents us with a novel difficulty: the presence for $-f^*$ of a maximum at $x = x_0$ gives rise (see Section 7.1.2 *infra*) to an inner generator Li located at a point $\omega'_F = e^{-\eta'_F}$ (with $\eta'_F = \int_0^{x_0} f(x)dx$) that is closer to the origin than both 1 (location of Lo_{in}) and ω_F (location of Lo_{out}). So the method of Section 5.1 for translating coefficient asymptotics into *nearest* singularity description no longer applies. One must then resort to a suitable *deformation* argument. We won't go into the details, but just mention a simplifying circumstance: from the fact that *inner generators* never produce *outer generators* (under alien differentiation), it follows that the actual manner of pushing Li beyond Lo_{in} and Lo_{out} (*i.e.* under right or left circumvention) doesn't matter.

[67] Recall that $f^{\vdash}(x) := -f(-x)$ and $f^{\vDash}(x) := -f(1-x)$.

7.1.2 From *original* to *inner*

Case 7.4 ($-f^*$ decreases on $[0, x_0]$, then increases on $[x_0, 1]$). When f has a simple zero at x_0, *i.e.* when the "tangency order"is $\kappa = 1$, we are back to the heuristics of Section 4.1. When f has a multiple zero (necessarily of odd order, if f^* is to have an extremum there), we have a tangency order $\kappa \in \{3, 5, 7\dots\}$ and the same argument as in Section 4.1 points to the existence of a singularity over η'_F in the ν-plane or ω'_F in the ζ-plane, with η'_F, ω'_F as above. In the ν-plane, this singularity is characterised by an upper-minor $\overset{\frown}{li}$ given by:

$$\overset{\frown}{li} := \mathrm{nir}({}^0 f) + \mathrm{nir}({}^0 f^{\vdash}) \quad \text{with} \quad {}^0 f(x) := f(x_0 + x). \tag{7.6}$$

In view of the parity relation (*cf.* Section 4.10) this implies:

$$\overset{\frown}{li}(\nu) = \sum_{0 \le k} h_{\frac{-\kappa+2k}{\kappa+1}} \nu^{\frac{-\kappa+2k}{\kappa+1}} \quad \text{with} \quad h = \mathrm{nir}(f) = \sum_{0 \le k} h_{\frac{-\kappa+k}{\kappa+1}} \nu^{\frac{-\kappa+k}{\kappa+1}}. \tag{7.7}$$

Thus, only every second coefficient of h goes into the making of $\overset{\frown}{li}$. Moreover, since κ here is necessarily odd, the ratio $\frac{-\kappa+2k}{\kappa+1}$ can never be an integer. This means that the corresponding *majors*[68] never carry any logarithms, but only fractional powers.

7.1.3 From *outer* to *inner* The relevant functional transform here is *nur*, which according to (5.13) is an infinite superposition of *nir* transforms applied separately to all determinations of $\log f(.)$. To calculate the alien derivatives of $\overset{\frown}{lo}_{\mathrm{in}}$ or $\overset{\frown}{lo}_{\mathrm{out}}$, we must therefore apply the recipe of the next para (Section 7.1.4) to the various $nir(2\pi i\, k + \log f(0) + \dots)$ or $nir(2\pi i\, k + \log(f(1) + \dots)$. Exceptionnally, if $2\pi i\, k + \log f(0)$ or $2\pi i\, k + \log f(1)$ vanishes for some k, we must also deal with tangency orders $\kappa > 0$ and apply the recipe of the para after next (Section 7.1.5). But in this as in that case, the result will always be *some* inner generator $\overset{\frown}{li}$, and never an outer one.

7.1.4 From *exceptional* to *inner* Let $\overset{\frown}{le}$ an exceptional generator with base point x_1. Assume, in other words, that $f(x_1) \neq 0$ and:

$$\overset{\frown}{le} = {}^{\nu_1} h = \mathrm{nir}({}^{x_1} f) \quad \text{with} \quad {}^{x_1} f(x) := f(x_1 + x), \quad \nu_1 := \int_0^{x_1} f(x)dx. \tag{7.8}$$

68 Whether $\overset{\smile}{li}$, $\overset{\vee}{li}$ or $\overset{\vee}{Li}$.

To calculate the alien derivatives of $\widehat{le}$, we go back to the long chain Section 4.2 and decompose the *nir*-transform into elementary steps from 1 to 7. The elementary Steps 1, 2, 4, 5, 7 neither produce nor destroy singularities. The steps that matter are the reciprocation (Step 3) and the *mir*-transform (Step 6). The singularities produced by reciprocation are easy to predict. As for the *mir*-transform, its integro-differential expression (4.39) and the properties of the Euler-Bernoulli numbers[69] show that the closest singularity or singularities of $\widehat{le}$ [70] necessarily correspond to closest singularity//ies of $\not{g}$ (see Lemma 4.7). Now comes the crucial, non-trivial fact: this one-to-one correspondance between singularities of $\not{g}$ and $\widehat{le}$ holds also in the large, at least when the initial input f is holomorphic. This is by no means obvious, since the singularities of $\not{g}$ might combine with those of β to produce infinitely many new ones, farther away, under the Hadamard product mechanism . To show that this *doesn't occur*, assume the existence of a point ν_2 in the ν-plane where ${}^{\nu_1}h = \widehat{le}$ is singular but g is regular. We can then write $\nu_2 = \int_0^{x_2} f(x)dx$ for some x_2 and then choose x_3 close enough to x_2 to ensure that the exceptional generator ${}^{\nu_3}h$ of base point x_3 is regular at ν_2.[71] We then use the bi-entireness of the finite *nir*-increment $\nabla h(\epsilon, \nu)$ with $\epsilon = x_3 - x_1$, $\nu = \nu_3 - \nu_1$ to conclude that ${}^{\nu_1}h := nir({}^{x_1}h)$, just like ${}^{\nu_3}h := nir({}^{x_3}h)$, is regular at ν_2.

7.1.5 From *inner* to *inner*. Ping-pong resurgence Let $\widehat{li}_1$ be an inner generator with base point x_1. This means that $f(x_1) = 0$ and:

$$\widehat{li}_1 = {}^{\nu_1}h = \mathrm{nir}({}^{x_1}f) \quad \text{with} \quad {}^{x_1}f(x) := f(x_1 + x), \quad \nu_1 := \int_0^{x_1} f(x)dx. \tag{7.9}$$

Assuming once again f to be holomorphic, the same argument as above shows that all singularities of $\widehat{li}_1$, not just the closest ones, correspond to zeros x_i of f. They are therefore inner generators $\widehat{li}_2, \widehat{li}_3, \widehat{li}_4 \ldots$ with base points $x_2, x_3, x_4 \ldots$ and the resurgence equations between them

$$\Delta_{\nu_q - \nu_p} \; \widehat{li}_p = \widehat{li}_q \tag{7.10}$$

[69] More exactly, the fact that the singularities of β are all on $2\pi i\mathbb{Z}^*$.

[70] *I.e.* those lying on the boundary of the disk of convergence. Recall that for an exceptional generator we have a tangency order $\kappa = 0$ and so $\widehat{le}$ is a regular, unramified germ at the origin.

[71] By ensuring that ${}^{\nu_3}h$ has ν_2 within its convergence disk.

will exactly mirror the resurgence equations between the singularities of g. The only difference is that if $\widehat{li}_p$ *sees* $\widehat{li}_q$, *i.e.* if (7.10) holds, then the converse is automatically true: $\widehat{li}_q$ *sees* $\widehat{li}_p$. Exceptional generators, on the other hand, *see* but are not *seen*.[72]

7.1.6 Recapitulation. One-way arrows, two-way arrows Let us sum up pictorially our findings for a holomorphic input f:

$$
\begin{array}{cccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \text{inner} & \leftarrow & \dots & \leftarrow \\
\uparrow & & & \nearrow & \uparrow\downarrow & & & \uparrow \\
\uparrow & & \text{outer} & & \uparrow\downarrow & & & \uparrow \\
\uparrow & \nearrow & & \searrow & \uparrow\downarrow & & & \uparrow \\
\text{original} & \rightarrow & \rightarrow & \rightarrow & \text{inner} & \leftarrow & \dots & \text{(exceptional)} \\
\downarrow & \searrow & & \nearrow & \uparrow\downarrow & & & \downarrow \\
\downarrow & & \text{outer} & & \uparrow\downarrow & & & \downarrow \\
\downarrow & & & \searrow & \uparrow\downarrow & & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \text{inner} & \leftarrow & \dots & \leftarrow
\end{array}
$$

The above picture displays four types of generators:

- one *original* generator, which is none other than the "cleansed" SP series;
- two *outer* generators (*in* and *out*) which may occasionally coalesce;
- a countable number of *inner* generators: as many as f has zeros;
- a continous infinity of *exceptional* generators: any x_i where f doesn't vanish can serve as base point.

The picture also shows five types of arrows linking these generators.[73] All these arrows are one-way, except for those linking pairs of inner generators.

As this "one-way/two-way traffic" suggests, the various generators differ widely as to origin, shape, and function.

[72] Regarding the inner generators, one may note that what matters is the geometry in the ν-plane, not in the x-plane. Consider for instance:

$$f(x) := (x - x_1)(x - x_2)(x - x_3) \quad \text{with} \quad x_1 = 1, x_2 = 2, x_3 = 2 + \epsilon + \epsilon^2 i \quad (0 < \epsilon \ll 1).$$

Then a simple calculation shows that the inner generator $\widehat{li}_1$ *sees* $\widehat{li}_2$ but not $\widehat{li}_3$, although x_1 *sees* x_2 and x_3. (On the other hand, $\widehat{li}_2$ *sees* both $\widehat{li}_1$ and $\widehat{l}\ i_3$).

[73] Meaning in each case that the *target* is generated by the *source* under alien differentiation.

The original generator clearly stands apart, not just because it kicks off the whole generation process, but also because it makes (immediate) sense only in the ζ-plane: in the ν-plane it is relegated to infinity.

Directly proceeding from it under the *nur*-transform, we have two outer generators, which in turn generate the potentially more numerous inner generators, this time under the *nir*-transform, relatively in each case to a given determination of $f = -\log F$. To each such determination (corresponding to an additive term $2\pi ik$) there answers a distinct inner algebra Inner_f spanned by K inner generators, with $K := \mathrm{card}\{f^{-1}(0)\}$.

Another way of entering the inner algebras is via exceptional generators, but these are "artificial" in the sense that they never occur naturally, *i.e.* under analytic continuation of the original generator. They are more in the nature of auxiliary tools.[74] Also, since each exceptional generator results from applying the *nir*-transform to a *given* determination of $f = \log F$, it gives acces to *one* inner algebra Inner_f, unlike the outer generators, which give access to *them all*.

These inner algebras Inner_f are in one-to-one correspondance with $\mathbb{Z}$. Though distinct (and usually disjoint) from each other, they are essentially isomorphic. Each of them is also "of one piece" in the sense that for any pair $\overset{\frown}{li}_p, \overset{\frown}{li}_q \in \mathrm{Inner}_f$, there is always a connecting chain li_{n_i} starting at $\overset{\frown}{li}_p$, ending at $\overset{\frown}{li}_q$, and such that any two neighbours li_{n_i} and $li_{n_{i+1}}$ *see* each other.

The emphasis so far has been on the singularities in the ν-plane. Those in the ζ-plane follow, except *over* the origin $\zeta = 0$, where quite specific and severe singularities may also occur (*at* the origin itself, *i.e.* on the main Riemann leaf, the SP function is of course regular). For a brief discussion of these 0-based singularities and their resurgence properties, see Section 7.2.1 below.

7.2 Meromorphic input F: the general picture

Let us briefly review the main changes which take place when we relax the hypothesis about $f := -\log(F)$ being holomorphic and simply demand that F be *meromorphic*.[75]

---

[74] As components of the *nur*-transform under the Poisson formula (see 5.13) and also, as we just saw, as mobile tools for sifting out true singularities from illusory ones (see Section 7.1.4).

[75] Since F and $F^{\vDash}$ (recall that $F^{\vDash}(x) := 1/F(1-x)$) are essentially on the same footing, it would make little sense to assume one to be *holomorphic* rather than the other. So we must assume *meromorphy*, even *strict* meromorphy, with at least one zero or pole.

7.2.1 Logarithmic/non-logarithmic singularities If F has at $x = 0$ a zero or pole of order $d \in \mathbb{Z}^*$, we must replace the $\sum_{0\leq m<n}$ summation in (1.2) by $\sum_{0<m<n}$ for the definition of the SP coefficients $J_F(n)$ to make sense. More significantly, depending of the parity of d, the outer and inner singularities may exchange their logarithmic/non-logarithmic nature. Recall that for the *cleansed* SP function and $d = 0$, the outer generators have purely logarithmic singularities[76] while the inner generators have power-type singularities, with strictly rational (non-entire) powers. That doesn't change when d is $\neq 0$, at least where the *cleansed* SP series are concerned. However, when we revert to the *raw* SP series, *i.e.* to the position prior to coefficient division by the ingress factor $Ig_F(n) \sim n^{-d/2}\,(c_0 + O(n^{-1}))$, we are faced with a neat dichotomy:

(i) d even: nothing changes;
(ii) d odd: everything gets reversed, with the outer singularities becoming strict rational (semi-integral) powers and the inner singularities becoming purely logarithmic.[77]

7.2.2 Welding the inner algebras into one The presence of even a single zero or pole in F, no matter where – whether at x=0 or x=1 or elsewhere – suffices to abolish the distinction between the various inner algebras $Inner_f$ attached to the various determinations of $f := -\log(F)$, since f itself now becomes multivalued and assumes the form:

$$f(x) = \sum d_i \,\log(x - x_i) + \text{holomorphic}(x). \tag{7.11}$$

Everything hinges on $d := g.c.d.(d_1, d_2, \ldots)$. If $d = 1$, then all inner algebras merge into one. If $d > 1$, they merge into d distinct but "isomorphic" copies.

Notice that no such change affects the outer generators, because these are constructed, not from f, but directly from F (in the case of the *raw* SP function) or $F^{1/2}$ (in the case of the *cleansed* SP function).

7.3 The ζ-plane and its violent 0-based singularities

Converting ν-singularities into ζ-singularities

So far, we have been describing the outer/inner singularities in the auxiliary ν-plane (more exactly, the ν-Riemann surface) which is naturally

[76] *I.e.* with majors of type $Reg_1(\zeta) + Reg_2(\zeta)\log(\zeta)$.

[77] At least in the generic case, *i.e.* for a tangency order $\kappa = 1$. For $\kappa > 1$, the inner singularities involve a mixture of rational powers and logarithms.

adapted to Taylor coefficient asymptotics. To revert to the original ζ-plane, we merely apply the formulas for Step 9 in the long chain of Section 4.2 which convert ν-singularities into ζ-singularities, for majors as well minors. The resurgence equations, too, carry over almost unchanged, with the additive indices ν_i simply turning into multiplicative indices ζ_i. But there is one exception, namely the origin $\zeta = 0$. Under the correspondence $\nu \mapsto \zeta = e^{\nu}$, the SP function's behaviour *over* $\zeta = 0$ will reflect its behaviour *over* the "point" $\Re(\nu) = -\infty$ on various Riemann leaves. This is the tricky matter we must now look into.

Description/expansion of the 0-based singularities

The SP function itself is regular *at* $\zeta = 0$, *i.e.* on the main Riemann leaf,[78] but usually not *over* $\zeta = 0$, *i.e.* on the other leaves. Studying these 0-singularities entirely reduces to studying the 0-singularities of the outer/inner generators $\overset{\wedge}{Lo}$ / $\overset{\wedge}{Li}$, which in turn reduces to investigating the ∞-behaviour of $\overset{\wedge}{lo}$ / $\overset{\wedge}{li}$. This can be done in the standard manner, by going to the long chain of Section 4.2 and applying the *mir*-transform to $\cancel{g}$, *but locally at* $-\infty$.[79] The integro-differential expansion for *mir* still converges in this case, but no longer formally so (*i.e.* coefficient-wise), and it still yields inner generators, but of a very special, quite irregular sort. Pulled back into the ζ-plane, they produce violent singularities *over* $\zeta = 0$, usually with exponentially explosive/implosive radial behaviour, depending on the sectorial neighbourhood of 0.

Resurgence properties of the 0-based singularities

Fortunately, no detailed *local* description of the 0-based singularities is required to calculate their alien derivatives and, therefore, to obtain a complete system of resurgence equations for our original SP function. Indeed, turning k times around $\zeta = 0$ on *some* leaf amounts to making a $2\pi i k$-shift in the ν-plane, again on *some* leaf. But the effect of that is easy to figure out, especially for an holomorphic input f (in that case, it simply takes us from one inner algebra $Inner_f$ to the next) but also for a general meromorphic input F (for illustrations, see the examples of Section 8.3, especially Examples 8.7 and 8.8. See also Section 6.6-8.).

[78] Or, if F has a zero/pole of odd order d, it is of the form $\zeta^{d/2}\varphi(\zeta)$, but again with a regular φ.

[79] Each time on the suitable leaf, of course.

7.4 Rational inputs F: the inner algebra

Let F be a rational function of degree d:

$$F(x) = \prod_{1\le j\le r} \left(1 - \frac{x}{\alpha_j}\right)^{d_j} \quad \text{with} \quad d_j \in \mathbb{Z}^*, \qquad (7.12)$$
$$\delta := \text{g.c.d.}(|d_1|, \ldots, |d_r|)$$

and let $x_0, \ldots, x_{d-1}$ be the zeros (counted with multiplicities) of the equation $F(x) = 1$. We then fix a determination of the the corresponding f:

$$f(x) = -\log(F) = -\sum_{1\le j\le r} d_j \log\left(1 - \frac{x}{\alpha_j}\right) \qquad (7.13)$$

with its Riemann surface S_f . We denote $X_j^n \subset S_f$ the set of all $x^\star \in S_f$ lying over $x_j \in \mathbb{C}$ and such that $f(x^\star) = 2\pi i\delta$ and select some point $x_0 \in X_0^0$ as base point of S_f. The internal generators will then correspond one-to-one to the points of $\cup_{1\le j<d} X_j^0$ and be located at points ν_j of the ramified ν-plane, with projections $\dot{\nu}_j$ such that:

$$\dot{\nu}_j - \dot{\nu}_i = \int_{x_i}^{x_j} f(x)dx \qquad (x_i \in X_i^0,\ x_j \in X_j^0). \qquad (7.14)$$

For two distinct points x'_j, x''_j in the same X_j^0, the above integral is obviously a multiple of $2\pi i\delta$. Therefore, three cases have to be distinguished.

Case 7.1. F has only one single zero α_1 (of any multiplicity $d_1 = p$) or again one single pole α_1 (of any multiplicity $d_1 = p$). In that case, we have exactly p sets X_j^0, but each one reduces to a single point, since f has only one single logaritmic singularity. That case ("monomial input F") was investigated in detail in Section 6.6, Section 6.7, Section 6.8.

Case 7.2. F has one simple zero α_1 and one simple pole α_2. The position is now the reverse: we then have only one set X_0^0,but with a countable infinity of points in it, since f has now two logarithmic singularities, thus allowing integrals (7.14) with distinct end points x'_0, x''_0 both in X_0^0. (See Section 8.11 below).

Case 7.3. F has either more than two distinct zeros, or more than two distinct poles, or both. We then have $p + q$ distinct sets X_j^0 , with p (respectively q) the number of distinct zeros (respectively poles). Each such X_j^0 contains a countable number of points x_j, to which there answer, in the ramified ν-plane, distinct singular points ν_j that generate a set $\mathcal{N}_j$

whose projection $\dot{\mathcal{N}}_j$ on $\mathbb{C}$ is of the form $\nu_j + \Omega$, with

$$\Omega = \left\{ \omega,\ \omega = \sum_{\sum n_j d_j = 0} n_j\, d_j\, \eta_j = -2\pi i \sum_{\sum n_j d_j = 0} n_j\, d_j\, \alpha_j \right\} \tag{7.15}$$

$$\eta_j = -\int_{\mathcal{I}_j} \log\left(1 - \frac{x}{\alpha_j}\right) dx = 2\pi i\, (x_0 - \alpha_j) \tag{7.16}$$

and with integration loops $\mathcal{I}_j$ so chosen as to generate the fundamental homotopy group of $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_r\}$. Each $\mathcal{I}_j$ describes a positive turn round α_j and the choice of the loops' common end-point is immaterial, since changing the end-point merely adds adds a common constant to each η_j, which constant cancels out from the sums ω due to the condition $\sum n_j\, d_j = 0$.

Though Ω usually fails to be discrete as soon as ≥ 3, the sets $\mathcal{N}_j$ are of course always discrete in the ramified ν-plane. In particular, from any given singular point $\nu_i \in \mathcal{N}_i$ only finitely many $\nu_j \in \mathcal{N}_j$ can be *seen* – those namely that correspond to *simple* integration paths in (7.14), *i.e.* typically paths whose projection on $\mathbb{C}$ is *short* and doesn't *self-intersect*.[80] The other points $\nu_j \in \mathcal{N}_j$ are located on more removed Riemann leaves and therefore *hidden from view* (from ν_i).

8 The inner resurgence algebra for SP series

8.1 Polynomial inputs f. Examples

Example 8.1 ($f(x) = x^r$). There is only one inner generator $\hat{h}(\nu)$ which, up to to the factor $\nu^{-1/2}$, is an entire function of ν.

Example 8.2 ($f(x) = x^r - 1$). There are r inner generators. We have exact radial symmetry, of radius 1 in the x-plane and radius $\eta = 1/(r+1)$ in the ν-plane. Every singular point there sees all the others: we have "multiple ping-pong", governed by a very simple resurgence system (see [15]).

Example 8.3 ($f(x) = \prod_{j=1}^{j=r}(x - x_j)$). Every such configuration, including the case of multiple roots, can be realised by continuous deformations of the radial-symmetric configuration of Example 8.2, and the thing is to

80 Various examples of such situations shall be given in Section 8.3, with *simple/complicated* integration paths corresponding to *visible/invisible* singularities. The general situation, with the exact criteria for visibility/invisibility, shall be investigated in [S.S.1].

keep track of the ν_j-pattern as the x_j-pattern changes. When $\nu_j - \nu_i$ becomes small while $x_j - x_i$ remains large, that usually reflects mutual invisibility of ν_i and ν_j. Thus, if $r = 3$ and $x_1 = 0, x_2 = 1, x_3 = 1+\epsilon\, e^{i\theta}$ with $0 < \epsilon \ll 1$, the case $\theta = \pi/2$ with its approximate symmetry $x_2 \leftrightarrow x_3$ corresponds to three mutually visible singularities ν_1, ν_2, ν_3, but when θ decreases to 0, causing x_3 to make a $-\pi/2$ rotation around x_2, the point ν_3 makes a $-3\pi/2$ rotation around ν_2, so that the projection $\dot{\nu}_3$ actually lands on the real interval $[\dot{\nu}_1, \dot{\nu}_2]$. But the new ν_3 has actually moved to an adjacent Riemann leaf and is no longer *visible* from ν_1.

8.2 Holomorphic inputs f. Examples

Example 8.4 ($f(x) = \exp(x)$). To the unique "zero" $x_0 = -\infty$ of $f(x)$ there answers a unique inner generator $\hat{h(\nu)}$. It is of rather exceptional type, in as far as its local behaviour is described by a *transseries* rather than a series, but the said transseries is still produced by the usual mechanism of the nine-link chain.

Example 8.5 ($f(x) = \exp(x) - 1$ or $f(x) = \sin^2(x)$). All zeros x_i of $f(x)$ contribute distinct inner generators, identical up to shifts but positioned at different locations ν_j.

Example 8.6 ($f(x)=\sin(x)$). Here, the periodic $f(x)$ still has infinitely many zeros but is constant-free (*i.e.* is itself the derivative of a periodic function). As a consequence, we have just two inner generators, at two distinct locations, like in the case $f(x) = 1 - x^2$ but of course with a more complex resurgence pattern.

8.3 Rational inputs F. Examples

Example 8.7 ($F(x) = (1 - x)$). The inner algebra here reduces to one generator $h(\nu)$ and a fairly trivial one at that, since $h(\nu) = const\, \nu^{-1/2}$, as given by the *semi-entire* part of the *nir*-transform. In contrast, the *entire* part of the *nir*-transform (which lacks intrinsic significance) is, even in this simplest of cases, a highly transcendental function: in particular, it verifies no linear ODE with polynomial coefficients.

Example 8.8 ($F(x) = (1 - x)^p$). Under the change $x \to p\,x$, this reduces the case of "monomial F", which was extensively investigated in Section 6.6, Section 6.7, Section 6.8. We have now exactly p internal generators $h_j(\nu)$ located at the unit roots $\nu_j = -e^{2\pi i\, j/p}$ and verifying a simple ODE of order p, with polynomial coefficients. Each singular point ν_j "sees" all the others, and the resurgence regimen is completely encapsulated in the matrices $\mathcal{M}_{p,q}$ of Section 6.7, which account for the basic closure phenomenon: a $4\pi i$-rotation (around any base point) leaves the whole picture unchanged.

Example 8.9 ($F(x) = \frac{x^2-\alpha^2}{1-\alpha^2} = \frac{x^2+\beta^2}{1+\beta^2}, \quad \alpha = i\,\beta$)**.** The general results of Section 7.4 apply here, with $x_0 = 0$, $x_1 = 1$ and the lattice $\Omega = 4\pi i\alpha\,\mathbb{Z} = -4\pi\beta\,\mathbb{Z}$. We have therefore two infinite series of internal generators in the ν-plane, located over $\dot\nu_0 + \Omega$ and $\dot\nu_1 + \Omega$ respectively, where the difference $\dot\nu_1 - \dot\nu_0$ may be taken equal to any determination of $-\int_0^1 \log(F(x))dx$. However, depending on the value of the parameters α, β, each singular point ν_j "sees" *one, two* or *three* singular points of the "opposite" series. Let us illustrate this on the three "real" cases:

Case 1: $0 < \beta$. The only singularity *seen* (respectively *half-seen*) from ν_0 is ν_1 (respectively ν_1^*) with

$$\begin{aligned} \dot\nu_1 &= \dot\nu_0 + 2\,\eta \\ \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 4\pi\beta \\ \text{with} \qquad \eta &= 2 - 2\beta\arctan(1/\beta) > 0. \end{aligned}$$

All other singularities above $\dot\nu_1 + \Omega$ lie are on further Riemann leaves. The singularity ν_1 corresponds to the straight integration path $\mathcal{I}_1$ whereas ν_1^* corresponds to either of the equivalent paths $\mathcal{I}_1^*$ and $\mathcal{I}_1^{**}$.

Case 2: $0 < \alpha < 1$. Only two singularities are *seen* from ν_0, namely ν_1^* and ν_1^{**} of projections:

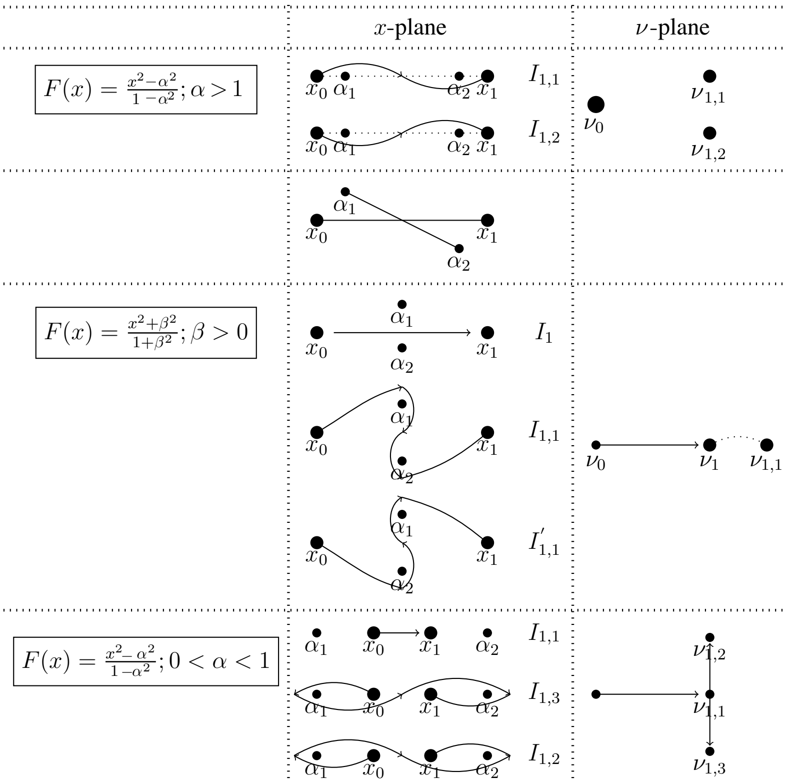
$$\begin{aligned} \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 2\,\pi i\alpha \\ \dot\nu_1^{**} &= \dot\nu_0 + 2\,\eta - 2\,\pi i\alpha \\ \text{with} \qquad \eta &= 2 - \alpha\log\left(\frac{1+\alpha}{1-\alpha}\right) > 0. \end{aligned}$$

They correspond to the integration paths $\mathcal{I}^*$ and $\mathcal{I}^{**}$.

Case 3: $1 < \alpha$. Three singularities are *seen* from ν_0, namely ν_1, ν_1^*, ν_1^{**} of projections:

$$\begin{aligned} \dot\nu_1 &= \dot\nu_0 + 2\,\eta \\ \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 2\,\pi i\alpha \\ \dot\nu_1^{**} &= \dot\nu_0 + 2\,\eta - 2\,\pi i\alpha \\ \text{with} \qquad \eta &= 2 - \alpha\log\left(\frac{\alpha+1}{\alpha-1}\right) < 0. \end{aligned}$$

They correspond to the integration paths $\mathcal{I}, \mathcal{I}^*, \mathcal{I}^{**}$.



Example 8.10 ($F(x) = \frac{x^p - \alpha^p}{1-\alpha^p} = \frac{x^p+\beta^p}{1+\beta^p}, \quad \epsilon = e^{\pi i/p}\beta$)**.** Here Ω is generated by the unit roots of order p. More precisely, due to the condition $\sum n_j\, d_j = 0$ in (219) (with $d_j = 1$ here) we have

$$\Omega = 2\pi i \alpha \Big((\epsilon - 1)\mathbb{Z} + (\epsilon^2 - 1)\mathbb{Z} + \ldots (\epsilon^{p-1} - 1)\mathbb{Z} \Big) \quad \text{with} \quad \epsilon := e^{2\pi i/p}.$$

Thus, except for $p \in \{2, 3, 4, 6\}$ the point set Ω is never discrete, but this doesn't prevent there being, from any point of the ramified ν-plane, only a finite number of *visible* singularities.

Case 1: $0 < \beta$.

$$\Omega_j = (\eta + \Omega)\,\epsilon^j = \eta\,\epsilon^j + \Omega \qquad (1 \le j \le p, \;\; \epsilon = e^{\pi i/p}\beta) \tag{8.1}$$

$$\eta = -p \sum_{1\le k} (-1)^k \frac{\beta^{-kp}}{kp+1} > 0 \qquad (\text{if } \; 1 \le \beta) \tag{8.2}$$

$$\eta = p - p\, b_p\, \beta - p \sum_{1 \le k} (-1)^k \frac{\beta^{kp}}{kp-1} > 0 \qquad (\text{if } \; 0 < \beta \le 1) \tag{8.3}$$

$$b_p = \int_0^1 \frac{1+t^{p-2}}{1+t^p} dt = 1 - 2 \sum_{1 \le k} \frac{(-1)^k}{k^2 p^2 - 1}. \tag{8.4}$$

Case 2: $0 < \alpha < 1$.

$$\begin{aligned}\Omega_j &= (\eta + \pi i \alpha + \Omega)\epsilon^j = (\eta - \pi i \alpha + \Omega)\epsilon^j = (\eta + \pi i \alpha)\epsilon^j + \Omega \\ &= (\eta - \pi i \alpha)\,\epsilon^j + \Omega \qquad (1 \le j \le p,\ \ \epsilon = e^{\pi i/p}\beta)\end{aligned} \tag{8.5}$$

$$\eta = p - p\,a_p\,\alpha - p\sum_{1\le k}\frac{\alpha^{kp}}{kp-1} > 0 \tag{8.6}$$

$$a_p = \int_0^1 \frac{1-t^{p-2}}{1-t^p}dt \ = 1 - 2\sum_{1\le k}\frac{1}{k^2p^2-1}. \tag{8.7}$$

Case 3: $1 < \alpha$.

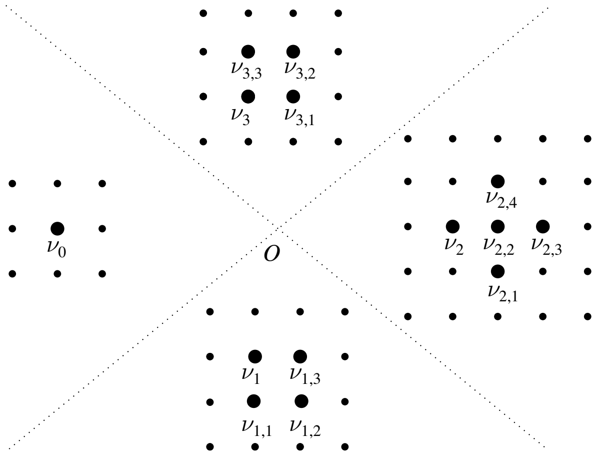
$$\Omega_j = (\eta + \Omega)\,\epsilon^j = \eta\,\epsilon^j + \Omega \quad (1 \le j \le p,\ \epsilon = e^{\pi i/p}\beta) \tag{8.8}$$

$$\eta = -p\sum_{1\le k}\frac{\alpha^{-kp}}{kp+1} < 0. \tag{8.9}$$

Remark. The expressions (8.4) for b_p are obtained by identifying the two distinct expressions (8.2), (8.3) for η which are are equally valid when $\beta = 1$. The expressions (8.7) for a_p are *formally* obtained in the same way, *i.e.* by equating the expressions (8.6), (8.9) when $\alpha = 1$, but since both diverge in that case, the derivation is illegitimate, and the proper way to proceed is by rotating α by $e^{\pi i/p}$ so as to fall back on the situation of case 1. Here are the $\mathbb{Z}$-irreducible equations verified by the first algebraic numbers $\alpha_p := \frac{p}{\pi}\,a_p$ and $\beta_p := \frac{p}{\pi}\,b_p$:

$$\begin{array}{ll}
0 = \alpha_2 & 0 = \beta_2 - 1 \\
0 = 3\,\alpha_3^2 - 1 & 0 = 3\,\beta_3^2 - 4 \\
0 = \alpha_4 - 1 & 0 = \beta_4^2 - 2 \\
0 = 5\,\alpha_5^4 - 10\,\alpha_5^2 + 1 & 0 = 5\,\beta_5^4 - 20\,\beta_5^2 + 16 \\
0 = \alpha_6^2 - 3 & 0 = \beta_6 - 2 \\
0 = \alpha_8^4 - 6\,\alpha_8^2 + 1 & 0 = \beta_8^4 - 8\,\beta_8^2 + 8 \\
0 = \alpha_{10}^4 - 10\,\alpha_{10}^2 + 5 & 0 = \beta_{10}^2 - 2\,\beta_{10} - 4 \\
0 = \alpha_{12}^2 - 4\,\alpha_{12} + 1 & 0 = \beta_{12}^4 - 16\,\beta_{12}^2 + 16.
\end{array}$$

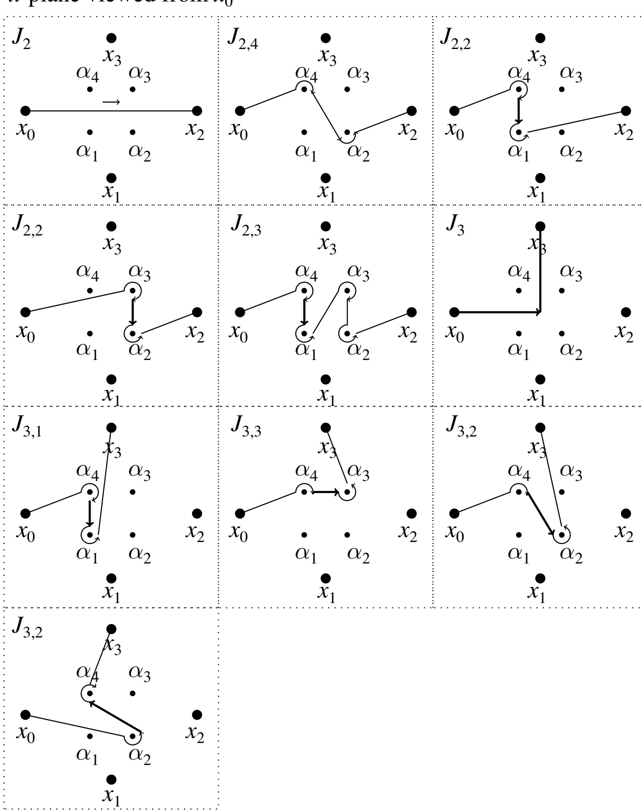
Let us illustrate the situation for $p = 4$ in all three “real” cases. We choose one singularity ν_0, corresponding to $x_0 = -1$ (respectively $x_0 = 1$) in Case 1 or 2 (respectively 3), as base point of the ν-plane, and plot as bold (respectively faint) points all singularities visible or semi-visible from ν_0 (respectively the closest invisible ones). For clarity, the scale (*i.e.* the relative values of α, η, π) has not been strictly respected – only the points’ relatives position has.

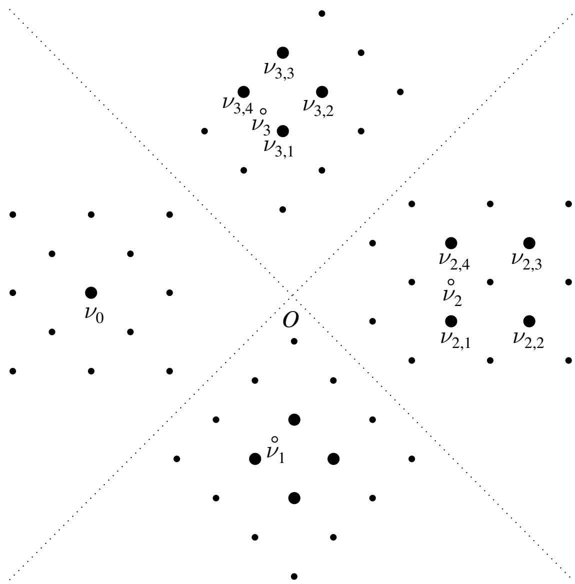


$$\boxed{F(x)=\frac{x^4+\beta^4}{1+\beta^4}\ ;\ \beta>0}$$

ν-plane viewed from ν_0

x-plane viewed from x_0

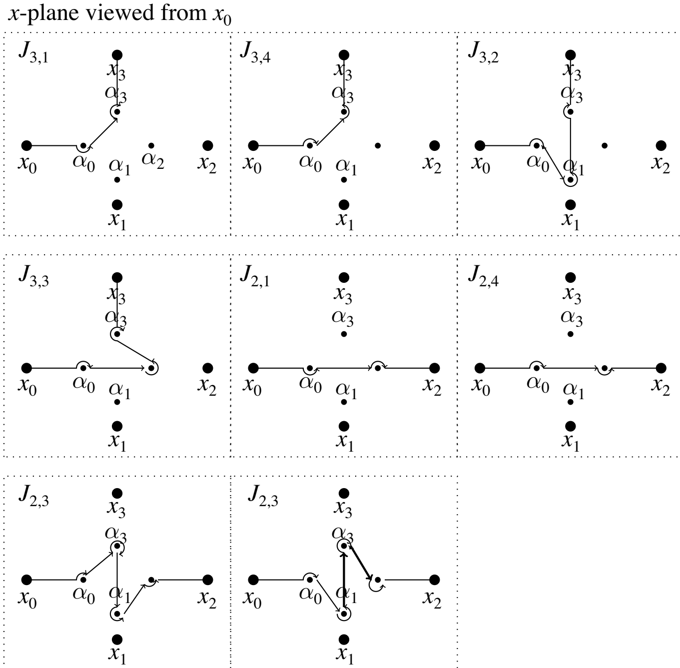


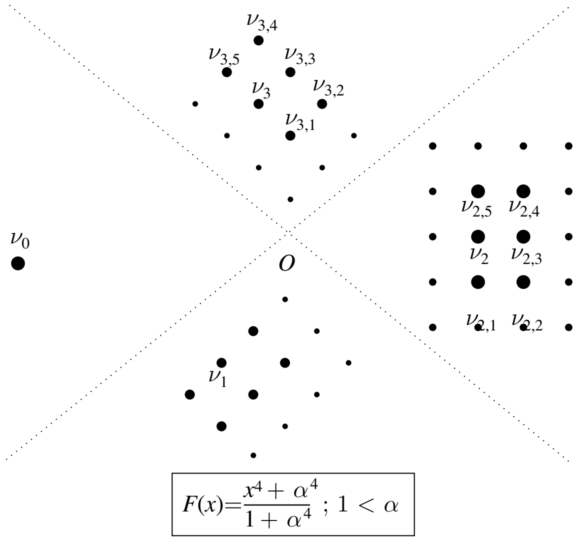


$$F(x)=\frac{x^4+\alpha^4}{1+\alpha^4}\ ;\ 0<\alpha<1$$

ν-plane viewed from ν_0

x-plane viewed from x_0





ν-plane viewed from ν_0

Example 8.11 ($F(x) = \frac{1-x}{1+x}$ **or** $F(x) = \frac{1-x/\alpha}{1+x/\beta}$**).** This interesting case is the only one where, despite the equation $F(x) = 1$ having only one solution $x_0 = 0$, the function f has two logarithmic singularities, so that we get a non-trivial set $\Omega = 2\pi i\mathbb{Z}$ and infinitely many copies of one and the same inner generator. From any given singularity ν_0 there are two *visible* neighbouring singularities over $\dot\nu_0 \pm 2\pi i$ and infinitely many *semi-visible* ones over $\dot\nu_0 \pm 2\pi i\, k$ $(k \geq 2)$.

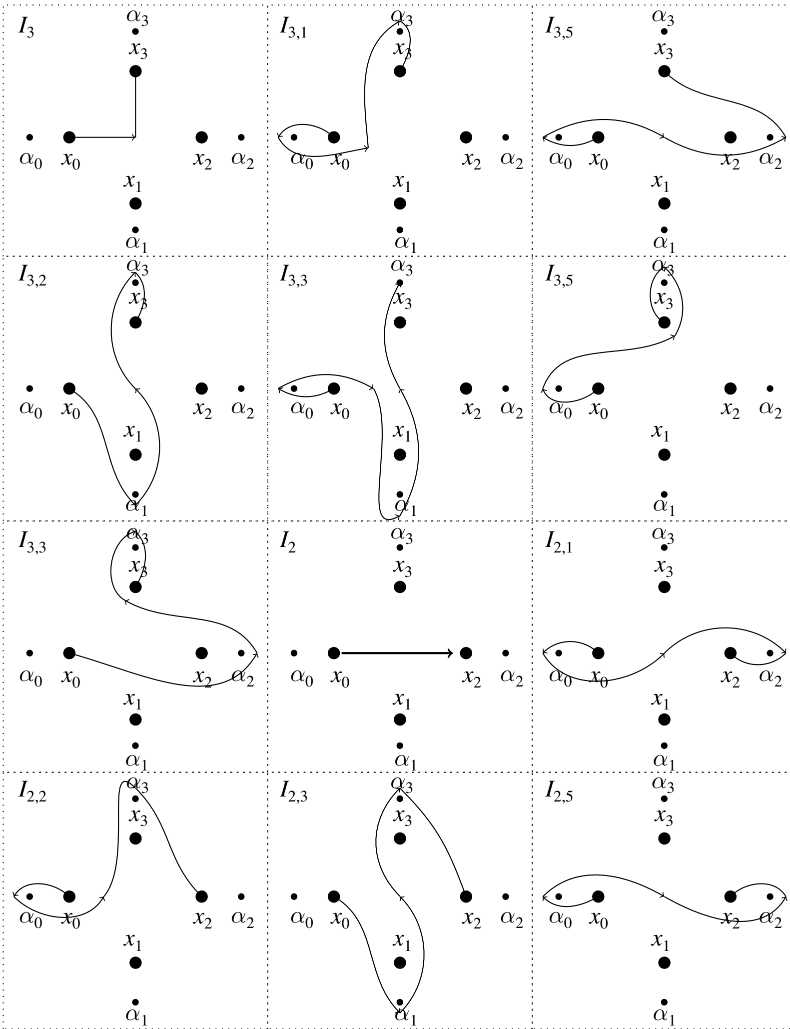
Example 8.12 ($F(x) = \frac{(1-x/\alpha_1)(1-x/\alpha_2)}{(1-x/\alpha_3)}$ **or** $F(x) = \frac{(1-x/\alpha_1)(1-x/\alpha_2)}{(1-x/\alpha_3)(1-x/\alpha_4)}$**).** Here Here the equation $F(x) = 0$ has two distinct solutions, so we have two distinct families of "parallel" inner generators, and a set Ω which is generically discrete in the first sub-case (no α_4) and generically non-discrete in the second sub-case.[81]

8.4 Holomorphic/meromorphic inputs F. Examples

Example 8.13 ($F = \prod_{j=1}^{\infty}\left(1 - \frac{x}{\alpha_j}\right) e^{A_j(x)}$ **or** $F = \frac{\prod_{j=1}^{\infty}(1-x/\alpha_j)e^{A_j(x)}}{\prod_{j=1}^{\infty}(1-x/\beta_j)e^{B_j(x)}}$**).** Predictably enough, we inherit here features from the case of polynomial

[81] As usual, this discrete/non-discrete dichotomy applies only to the projection on $\mathbb{C}$ of the ramified ν-plane, which is itself always a *discrete* Riemann surface, with only a *discrete* configuration of singular points visible from any given base point.

x-plane viewed from x_0



inputs f and from that of rational inputs F, but three points need to be stressed:

(i) the presence of even a single zero α_j or of a single pole β_j is enough to weld all inner algebras Inner_f into one (see Section 7.2.2 *supra*);

(ii) though, for a given x_0, the numbers $\eta_{0,j} := -\int_{x_0}^{x_j} f(x)dx$ may accumulate 0, the corresponding singularities ν_j never accumulate ν_0 in the ramified ν-plane;

(iii) the question of deciding which integration paths (in the x-plane) lead to visible singularities (in the ν-plane) is harder to decide than for purely polynomial inputs f or purely rational inputs F because, unlike in these two earlier situations, we don't always have the option of *deforming* a configuration with full radial symmetry. The precise criteria for visibility/invisibility shall be given in [16].

Example 8.14 (F = trigonometric polynomial). The series associated with knots tend to fall into this class. Significant simplifications occur, especially when $f = -\log(F)$ is itself the derivative of a periodic function, because the number of singularities ν_j then becomes finite up to Ω-translations. The special case $F(x) = 4\sin^2(\pi x)$, which is relevant to the knot 4_1, is investigated at length in the next section.

9 Application to some knot-related power series

9.1 The knot 4_1 and the attached power series G^{NP}, G^P

Knot theory attaches to each knot $\mathcal{K}$ two types of power series: the so-called *non-perturbative* series $G^{NP}_{\mathcal{K}}$ and their *perturbative* companions $G^P_{\mathcal{K}}$. Both encode the bulk of the invariant information about $\mathcal{K}$ and both are largely equivalent, though non-trivially so: each one can be retieved from the other, either by non-trivial arithmetic manipulations (the Habiro approach) or under a non-trivial process of analytic continuation (the approach favoured in this section).

The main ingredient in the construction of $G^{NP}_{\mathcal{K}}$ and $G^P_{\mathcal{K}}$ is the so-called *quantum factorial*, classically denoted $(q)_m$:

$$(q)_m := \prod_{k=1}^{k=m} (1-q^k). \tag{9.1}$$

For the simplest knots, namely $\mathcal{K} = 3_1$ or 4_1 in standard notation, the general definitions yield:

$$\begin{aligned}
\Phi_{3_1}(q) &:= \sum_{m\geq 1} (q)_m & \Phi_{4_1}(q) &:= \sum_{m\geq 1} (q)_m (q^{-1})_m \\
\hat{G}{}^{NP}_{3_1}(\zeta) &:= \sum_{n\geq 0} \Phi_{3_1}(e^{2\pi i/n})\,\zeta^n & \hat{G}{}^{NP}_{4_1}(\zeta) &:= \sum_{n\geq 0} \Phi_{4_1}(e^{2\pi i/n})\,\zeta^n \\
\tilde{G}^P_{3_1}(n) &:= \Phi_{3_1}(e^{-1/n}) = \sum c_k\, n^{-k} & \tilde{G}^P_{4_1}(n) &:= \Phi_{4_1}(e^{-1/n}) = \sum c^*_k\, n^{-k} \\
\hat{G}{}^{P}_{3_1}(\nu) &:= \sum c_k \frac{\nu^{k-1}}{(k-1)!} & \hat{G}{}^{P}_{4_1}(\nu) &:= \sum c^*_k \frac{\nu^{k-1}}{(k-1)!}.
\end{aligned}$$

A few words of explanation are in order here.

First: when we plug unit roots $q = e^{2\pi i/n}$ into the infinite series $\Phi_{3_1}(q)$ or $\Phi_{4_1}(q)$, these reduce to finite sums.

Second: the coefficients $\Phi_{3_1}(e^{2\pi i/n})$ or $\Phi_{4_1}(e^{2\pi i/n})$ thus defined are syntactically of *sum-product* type, relative to the driving functions:

$$\begin{aligned} F_{3_1}(x) &:= 1 - e^{2\pi i x}; \\ F_{4_1}(x) &:= (1 - e^{2\pi i x})(1 - e^{-2\pi i x}) \;=\; 4\sin^2(\pi x). \end{aligned} \tag{9.2}$$

Third: whereas the non-perturbative series $\hat{G}{}^{NP}$ clearly possess positive radii of convergence, their perturbative counterparts $\tilde{G}^P$ are divergent power series of $1/n$, of Gevrey type 1, *i.e.* with coefficients bounded by

$$|c_k| < \text{Const}\, k!, \qquad |c_k^*| < \text{Const}^*\, k!$$

Fourth: the perturbative series $\tilde{G}^P(n)$ being Gevrey-divergent, we have to take their Borel transforms $\hat{G}{}^P(\nu)$ to restore convergence.

Here, we won't discuss the series attached to knot 3_1, because that case has already been thoroughly investigated by Costin-Garoufalidis [3, 4] and also because it is rather atypical, with an uncharacteristically poor resurgence structure: indeed, $G_{3_1}^{NP}$ and $G_{3_1}^{P}$ give rise to only *one* inner generator Li, whereas it takes at least two of them for the phenomenon of ping-pong resurgence to manifest.

So we shall concentrate on the next knot, to wit 4_1, with its driving function $F(x) := 4\sin^2(\pi x)$. That case was/is also being investigated by Costin-Garoufalidis but with methods quite different from ours: see Section 12.2 below for a comparison. Here, we shall approach the problem as a special case of *sum-product* series, unravel the underlying resurgence structure, and highlight the typical interplay between the four types of generators: *original, exceptional, outer, inner*.

Our main *original* generator Lo and main *outer* generator Lu, both corresponding to the same base point $x = 0$, shall turn out to be essentially equivalent, respectively, to the *non-perturbative* and *perturbative* series of the classical theory, with only minor differences stemming from the *ingress factor* (see below) and a trivial $2\pi i$ rotation. The exact correspondence goes like this:

$$\hat{G}{}_{4_1}^{NP}(\zeta) \equiv \zeta\,\partial_\zeta\,\overset{\wedge}{Lo}\,(\zeta) \tag{9.3}$$

$$\hat{G}{}_{4_1}^{P}(\nu) \equiv \frac{1}{2\pi i}\,\partial_\zeta\,\overset{\wedge}{lu}\,(2\pi i\nu) = \frac{1}{2\pi i}\,\partial_\zeta\,\overset{\wedge}{Lu}\,(e^{2\pi i\nu} - 1). \tag{9.4}$$

But we shall also introduce other generators, absent from the classical picture: namely an *exceptional* generator Le, relative to the base-point

$x = 1/2$, as well as a new pair consisting of a secondary *original* generator Loo and a secondary *outer* generator Luu, also relative to the base-point $x = 1/2$.

We shall show that these generators don't self-reproduce under alien differentiation, but vanish without trace: they are mere *gates* for entering the true core of the resurgence algebra, namely the *inner algebra*, which in the present instance will be spanned by just two *inner* generators, Li and Lii.

9.2 Two contingent ingress factors

Applying the rules of Section 3 we find that to the driving function *Fo* and its translate *Foo*:

$$\begin{aligned} Fo(x) &= F(x) = 4\sin^2(\pi\, x); \\ Foo(x) &= F\left(x + \frac{1}{2}\right) = 4\cos^2(\pi\, x) \end{aligned} \tag{9.5}$$

there correspond the following ingress factors:

$$Ig_{Fo}(n) := (4\, pi^2)^{-1/2}\ (2\pi\, n)^{2/2} = n; \qquad Ig_{Foo}(n) := 4^{1/2} = 2. \tag{9.6}$$

Their elementary character stems from the fact the only contributing factors in $Fo(x)$ and $Foo(x)$ are $4\pi^2 x^2$ and 4 respectively. All other *binomial* or *exponential* factors inside $Fo(x)$ and $Foo(x)$ contribute nothing, since they are *even* functions of x.

Leaving aside the totally trivial ingress factor $Ig_{Foo}(n) = 2$, we can predict what the effect will be of removing $Ig_{Fo}(n) = n$ from $\hat{G}{}^{NP}(\zeta)$ and all its alien derivatives: it will *smoothen* all singularities under what shall amount to one ζ-integration. In particular, it shall replace the leading terms $C_1(\zeta - \zeta_1)^{-5/2}$ and $C_3(\zeta - \zeta_3)^{-5/2}$ in the singularities of $\hat{G}{}^{NP}(\zeta)$ at ζ_1 and ζ_3 by the leading terms $C_1'(\zeta - \zeta_1)^{-3/2}$ and $C_3'(\zeta - \zeta_3)^{-3/2}$ typical of *inner generators* produced by driving functions $f(x)$ of tangency order $m = 1$ (see Section 4).

Remark. An alternative, more direct but less conceptual way of deriving the form of the ingress factor $Ig_{Fo}(n) = n$ would be to use the following trigonometric identities:

$$K_{n,n-1} \equiv n^2,\ \ K_{2\,n,n-1} \equiv n,\ \ K_{2\,n,n} \equiv 4\,n,\ \ K_{2\,n+1,n} \equiv 2\,n + 1 \tag{9.7}$$

with

$$K_{n,m} := \prod_{1\le k\le m} F\left(\frac{k}{m}\right) = 4^m \prod_{1\le k\le m} \sin^2\left(\pi\frac{k}{m}\right). \tag{9.8}$$

9.3 Two original generators Lo and Loo

Here are the power series of *sum-product* type corresponding to the driving functions Fo and Foo (mark the lower summation bounds: first 1, then 0):

$$\overset{\wedge}{Jo}\,(\zeta):=\sum_{1\leq n} Jo_n\,\zeta^n \quad \text{with} \quad Jo_n := \sum_{m=1}^{m=n}\prod_{k=1}^{k=m} Fo\left(\frac{k}{n}\right) \tag{9.9}$$

$$\overset{\wedge}{Joo}\,(\zeta):=\sum_{1\leq n} Joo_n\,\zeta^n \quad \text{with} \quad Joo_n := \sum_{m=0}^{m=n}\prod_{k=0}^{k=m} Foo\left(\frac{k}{n}\right). \tag{9.10}$$

After removal of the respective ingress factors $Ig_{Fo}(n)$ and $Ig_{Foo}(n)$ these become our two *original generators*:

$$\overset{\wedge}{Lo}\,(\zeta):=\sum_{1\leq n} Lo_n\,\zeta^n=\sum_{1\leq n}\frac{1}{n}Jo_n\zeta^n \Longrightarrow \overset{\wedge}{Lo}\,(\zeta)=\int_0^\zeta \overset{\wedge}{Jo}(\zeta')\frac{d\zeta'}{\zeta'} \tag{9.11}$$

$$\overset{\wedge}{Loo}\,(\zeta):=\sum_{1\leq n} Loo_n\,\zeta^n=\sum_{1\leq n}\frac{1}{2}Joo_n\,\zeta^n \Longrightarrow \overset{\wedge}{Loo}\,(\zeta)=\frac{1}{2}\,\overset{\wedge}{Joo}\,(\zeta). \tag{9.12}$$

9.4 Two outer generators Lu and Luu

The two outer generators $\overset{\wedge}{Lu}$ and $\overset{\wedge}{Luu}$ (respectively their variants $\overset{\wedge}{\ell u}$ and $\overset{\wedge}{\ell uu}$) are produced as outputs H (respectively $\hat{k}$) by inputting $F = Fo$ or $F = 1/Foo$ into the short chain Section 5.2 and duly removing the ingress factor Ig_{Fo} or Ig_{Foo}. Since both $Fo(x)$ and $Foo(x)$ are even functions of x, we find:

$$\begin{array}{ll}
\tilde{\ell u}(n) :=\dfrac{1}{n}\displaystyle\sum_{1\leq m}\prod_{1\leq k\leq m} Fo\left(\frac{k}{n}\right) & \tilde{\ell uu}(n) :=\dfrac{1}{2}\displaystyle\sum_{1\leq m}\prod_{1\leq k\leq m} (1/Foo)\left(\frac{k}{n}\right)\\
\Downarrow & \Downarrow\\
\tilde{\ell u}(n) :=\displaystyle\sum_{1\leq k} c_{2k+1}\,n^{-2k-1} & \tilde{\ell uu}(n) :=\displaystyle\sum_{0\leq k} c^*_{2k}\,n^{-2k}\\
\Downarrow & \Downarrow\\
\overset{\wedge}{\ell u}\,(\nu) :=\displaystyle\sum_{1\leq k} c_{2k+1}\,\frac{\nu^{2k}}{(2k)!} & \overset{\wedge}{\ell uu}\,(\nu) :=\displaystyle\sum_{0\leq k} c^*_{2k}\,\frac{\nu^{2k-1}}{(2k-1)!}\\
\Downarrow & \Downarrow\\
\overset{\wedge}{Lu}\,(\zeta):=\overset{\wedge}{\ell u}\,(\log(1+\zeta)) & \overset{\wedge}{Luu}\,(\zeta):=\overset{\wedge}{\ell uu}\,(\log(1+\zeta)).
\end{array}$$

There is a subtle difference between the two columns, though. Whereas in the left column, the sum product $\sum\prod$ truncated at order m yields the *exact* values of all coefficients c_{2k+1} up to order m, the same doesn't hold true for the right column: here, the truncation of $\sum\prod$ at order m yields only approximate values of the coefficients c_{2k} (of course, the larger m, the better the approximation). This is because $Fo(0) = 0$ but $1/Foo(0) \neq 0$. Therefore, whereas the short, four-link chain of Section 5.2 suffices to give the exact coefficients c_{2k+1}, one must resort to the more complex *nur*-transform, as articulated in the long, nine-link chain of Section 5.2, to get the exact value of any given coefficient c^*_{2k}.

9.5 Two inner generators Li and Lii

The two outer generators $\overset{\wedge}{Li}$ and $\overset{\wedge}{Lii}$ (respectively their variants $\overset{\wedge}{\ell i}$ and $\overset{\wedge}{\ell ii}$) are produced as outputs h (respectively H) by inputting

$$\begin{aligned} f(x) = fi(x) &= -\log\left(4\sin^2\left(\pi\left(x+\frac{5}{6}\right)\right)\right) \\ &= +2\sqrt{3}\pi\, x + 4\pi^2 x^2 + O(x^3) \end{aligned} \tag{9.13}$$

$$\begin{aligned} f(x) = fii(x) &= -\log\left(4\sin^2\left(\pi\left(x+\frac{1}{6}\right)\right)\right) \\ &= -2\sqrt{3}\pi\, x + 4\pi^2 x^2 + O(x^3) \end{aligned} \tag{9.14}$$

into the long chain of Section 4.2 expressive of the *nir*-transform. However, due to an obvious symmetry, it is enough to calculate $\overset{\wedge}{\ell i}\ (\nu)$ and then deduce $\overset{\wedge}{\ell ii}\ (\nu)$ under (essentially) the chance $\nu \to -\nu$. Notice that the tangency order here is $m = 1$, leading to semi-integral powers of ν:

$$\overset{\wedge}{\ell i}\ (\nu) := \sum_{0\le n} d_{-\frac{3}{2}+n}\, \nu^{-\frac{3}{2}+n}; \quad \overset{\wedge}{\ell ii}\ (\nu) := \sum_{0\le n} (-1)^n\, d_{-\frac{3}{2}+n}\, \nu^{-\frac{3}{2}+n}. \tag{9.15}$$

Notice, too, that there is no need to bother about the ingress factors here: the very definition of the *nir*-transform automatically provides for their removal.

9.6 One exceptional generator Le

The exceptional generators $\overset{\wedge}{Le}$ (respectively their variant $\overset{\wedge}{\ell e}$) is produced as output h (respectively H) by inputting

$$\begin{aligned} f(x) = fo(x) &= -\log\left(4\sin^2\left(\pi\left(x+\frac{1}{2}\right)\right)\right) \\ &= -2\log 2 + \pi^2 x^2 + O(x^4) \end{aligned} \tag{9.16}$$

into the long chain of Section 4.2 expressive of the *nir*-transform. The tangency order here being $m = 0$ and $fo(x)$ being an even function of x, the series $\overset{\frown}{\ell e}$ (respectively $\overset{\wedge}{\ell e}$) carries only integral-even (respectively integral-odd) powers of ν:

$$\overset{\frown}{\ell e}\,(\nu) := \sum_{0 \leq n} c^{**}_{2n}\; \nu^{2n} \qquad \overset{\wedge}{\ell e}\,(\nu) := \sum_{0 \leq n} 2n\, c^{**}_{2n}\; \nu^{2n-1}. \tag{9.17}$$

As with the *inner generators*, the *nir*-transform automatically takes care of removing the ingress factor.

9.7 A complete system of resurgence equations

Before writing down the exact resurgence equations, let us depict them graphically, in the two pictures below, where each arrow connecting two generators signals that the *target generator* can be obtained as an alien derivative of the *source generator*.

$$
\begin{array}{ccccccc}
 & & \longrightarrow & \longrightarrow & \text{inner generator } Li & & \\
 & \nearrow & & \nearrow & \uparrow\downarrow & \nwarrow & \\
\text{original generator } Lo & \longrightarrow & \text{outer generator } Lu & & \uparrow\downarrow & & \text{exceptional generator } Le \\
 & & & \searrow & \uparrow\downarrow & \swarrow & \\
 & & & & \text{inner generator } Lii & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & \longrightarrow & \longrightarrow & \text{inner generator } Li & & \\
 & \nearrow & & \nearrow & \uparrow\downarrow & \nwarrow & \\
\text{original generator } Loo & \longrightarrow & \text{outer generator } Luu & & \uparrow\downarrow & & \text{exceptional generator } Le \\
 & & & \searrow & \uparrow\downarrow & \swarrow & \\
 & & & & \text{inner generator } Lii. & &
\end{array}
$$

We observe that whereas each inner generator is both *source* and *target*, the other generators (– original, outer, exceptional –) are *sources* only. Moreover, although there is perfect symmetry between Li and Lii within the inner algebra, that symmetry breaks down when we adduce the original generators *Lo* or *Loo*: indeed, Li is a target for both *Lo* and *Loo*, but its counterpart Lii is a target for neither.[82] Altogether, we get the six resurgence algebras depicted below, with the inner algebra as their

[82] At least, under *strict alien derivation*: this doesn't stand in contradiction to the fact that under *lateral* continuation (upper or lower) of *Lo* or *Loo* along the real axis, singularities $\pm 4\,i\,Lii$ can be "seen" over the point ζ_3. See Section 9.8.3 below.

common core:

inner algebra

$$\begin{array}{lclcl}
\{Li, Lii\} & \subset & \{Li, Lii, Lu\} & \subset & \{Li, Lii, Lu, Lo\} \\
\{Li, Lii\} & \subset & \{Li, Lii, Luu\} & \subset & \{Li, Lii, Luu, Loo, \} \\
\{Li, Lii\} & \subset & \{Li, Lii, Le\}. & &
\end{array}$$

Next, we list the points ζ_i where the singularities occur in the $zeta$-plane, and their real logarithmic counterparts ν_i in the ν-plane.

$$\begin{aligned}
\nu_0 &:= -\infty \\
\nu_1 &:= \int_0^{1/6} f(x)dx = -\frac{Li_2(e^{2\pi i/6}) - Li_2(e^{-2\pi i/6})}{2\pi i} = -0.3230659470\dots \\
\nu_2 &:= 0 \\
\nu_3 &:= \int_0^{5/6} f(x)dx = +\frac{Li_2(e^{2\pi i/6}) - Li_2(e^{-2\pi i/6})}{2\pi i} = +0.3230659470\dots \\
\zeta_0 &:= 0 \\
\zeta_1 &:= \exp(\nu_1) = 0.723926112\dots = 1/\zeta_3 \\
\zeta_2 &:= 1 \\
\zeta_3 &:= \exp(\nu_3) = 1.381356444\dots
\end{aligned}$$

The assignment of *generators* to *singular points* goes like this:[83]

$$\begin{array}{lllll}
\overset{\wedge}{Lo} \text{ and } \overset{\wedge}{Loo} & \text{at } \zeta_0 & & & \\
\overset{\wedge}{Li} & \text{at } \zeta_1 \ ; & \underline{\overset{\wedge}{Li}} & \text{at} & \underline{\zeta_1} \\
\overset{\wedge}{Lu} \text{ and } \overset{\wedge}{Luu} \text{ and } \overset{\wedge}{Le} & \text{at } \zeta_2 \ ; & \underline{\overset{\wedge}{Luu}} & \text{at} & \underline{\zeta_2} \\
\overset{\wedge}{Lii} & \text{at } \zeta_3 \ ; & \underline{\overset{\wedge}{Lii}} & \text{at} & \underline{\zeta_3}
\end{array}$$

with

$$\underline{\zeta_1} := -\zeta_1, \quad \underline{\zeta_2} := -\zeta_2, \quad \underline{\zeta_3} := -\zeta_3$$

and

$$\underline{\overset{\wedge}{Li}}(\zeta) := \overset{\wedge}{Li}\ (-\zeta), \quad \underline{\overset{\wedge}{Lii}}(\zeta) := \overset{\wedge}{Lii}\ (-\zeta), \quad \underline{\overset{\wedge}{Luu}}(\zeta) := \overset{\wedge}{Luu}\ (-\zeta).$$

83 In the ζ-plane, for definiteness.

The correspondence between singularities in the ζ- and ν-planes is as follows:

minors ζ plane	minors ν plane	majors ζ plane	majors ν plane
$\overset{\wedge}{Li}(\zeta)$	$= \overset{\wedge}{li}\big(\log(1+\zeta/\zeta_i)\big)$	$\overset{\vee}{Li}(\zeta)$	$= \overset{\vee}{li}\big(-\log(1-\zeta/\zeta_i)\big)$
$\overset{\wedge}{Lii}(\zeta)$	$= \overset{\wedge}{lii}\big(\log(1+\zeta/\zeta_{ii})\big)$	$\overset{\vee}{Lii}(\zeta)$	$= \overset{\vee}{lii}\big(-\log(1-\zeta/\zeta_{ii})\big)$
$\overset{\wedge}{Lu}(\zeta)$	$= \overset{\wedge}{lu}\big(\log(1+\zeta)\big)$	$\overset{\vee}{Lu}(\zeta)$	$= \overset{\vee}{lu}\big(-\log(1-\zeta)\big)$
$\overset{\wedge}{Luu}(\zeta)$	$= \overset{\wedge}{luu}\big(\log(1+\zeta)\big)$	$\overset{\vee}{Luu}(\zeta)$	$= \overset{\vee}{luu}\big(-\log(1-\zeta)\big)$
$\overset{\wedge}{Le}(\zeta)$	$= \overset{\wedge}{le}\big(\log(1+\zeta)\big)$.		

With all these notations and definitions out of the way, we are now in a position to write down the resurgence equations connecting the various generators:

Resurgence algebra generated by Lo

$$\Delta_{\zeta_1}\overset{\lozenge}{Lo} = 2\,\overset{\lozenge}{Li} \qquad \Delta_{\zeta_3-\zeta_2}\overset{\lozenge}{Lu} = \tfrac{2}{2\pi}\,\overset{\lozenge}{Lii} \qquad \Delta_{\zeta_3-\zeta_1}\overset{\lozenge}{Li} = \tfrac{3}{2\pi}\,\overset{\lozenge}{Lii}$$

$$\Delta_{\zeta_2}\overset{\lozenge}{Lo} = 1\,\overset{\lozenge}{Lu} \qquad \Delta_{\zeta_1-\zeta_2}\overset{\lozenge}{Lu} = \tfrac{2}{2\pi}\,\overset{\lozenge}{Li} \qquad \Delta_{\zeta_1-\zeta_3}\overset{\lozenge}{Lii} = \tfrac{3}{2\pi}\,\overset{\lozenge}{Li}$$

$$\Delta_{\zeta_3}\overset{\lozenge}{Lo} = 0\,\overset{\lozenge}{Lii}.$$

Resurgence algebra generated by Loo

$$\Delta_{\zeta_1}\overset{\lozenge}{Loo} = 2\,\overset{\lozenge}{Li} \qquad \Delta_{\zeta_3-\zeta_2}\overset{\lozenge}{Luu} = \tfrac{2}{2\pi}\,\overset{\lozenge}{Lii} \qquad \Delta_{\zeta_3-\zeta_1}\overset{\lozenge}{Li} = \tfrac{3}{2\pi}\,\overset{\lozenge}{Lii}$$

$$\Delta_{\zeta_2}\overset{\lozenge}{Loo} = 1\,\overset{\lozenge}{Luu} \qquad \Delta_{\zeta_1-\zeta_2}\overset{\lozenge}{Luu} = -\tfrac{2}{2\pi}\,\overset{\lozenge}{Li} \qquad \Delta_{\zeta_1-\zeta_3}\overset{\lozenge}{Lii} = \tfrac{3}{2\pi}\,\overset{\lozenge}{Li}$$

$$\Delta_{\zeta_3}\overset{\lozenge}{Loo} = 0\,\overset{\lozenge}{Lii}$$

$$\Delta_{\underline{\zeta_1}}\overset{\lozenge}{Loo} = 0\,\underline{\overset{\lozenge}{Li}} \qquad \Delta_{\underline{\zeta_3}-\underline{\zeta_2}}\underline{\overset{\lozenge}{Luu}} = \tfrac{2}{2\pi}\,\underline{\overset{\lozenge}{Lii}} \qquad \Delta_{\underline{\zeta_3}-\underline{\zeta_1}}\underline{\overset{\lozenge}{Li}} = \tfrac{3}{2\pi}\,\underline{\overset{\lozenge}{Lii}}$$

$$\Delta_{\underline{\zeta_2}}\overset{\lozenge}{Loo} = -2\,\underline{\overset{\lozenge}{Luu}} \qquad \Delta_{\underline{\zeta_1}-\underline{\zeta_2}}\underline{\overset{\lozenge}{Luu}} = -\tfrac{2}{2\pi}\,\underline{\overset{\lozenge}{Li}} \qquad \Delta_{\underline{\zeta_1}-\underline{\zeta_3}}\underline{\overset{\lozenge}{Lii}} = \tfrac{3}{2\pi}\,\underline{\overset{\lozenge}{Li}}$$

$$\Delta_{\underline{\zeta_3}}\overset{\lozenge}{Loo} = 0\,\underline{\overset{\lozenge}{Lii}}.$$

Resurgence algebra generated by Le

$$\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Le} = \tfrac{2}{2\pi}\,\overset{\diamond}{Lii} \qquad \Delta_{\zeta_3-\zeta_1}\overset{\diamond}{Li} = \tfrac{3}{2\pi}\,\overset{\diamond}{Lii}$$

$$\Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Le} = -\tfrac{2}{2\pi}\,\overset{\diamond}{Li} \qquad \Delta_{\zeta_1-\zeta_3}\overset{\diamond}{Lii} = \tfrac{3}{2\pi}\,\overset{\diamond}{Li}.$$

9.8 Computational verifications

In order to check numerically our dozen or so resurgence equations, we shall make systematic use of the method of Section 2.3 which describes *singularities* in terms of *Taylor coefficient asymptotics*. Three situations, however, may present themselves:

(i) the singularity under investigation is closest to zero. This is the most favourable situation, as it makes for a straightforward application of Section 2.3;
(ii) the singularity under investigation is not closest to zero, but becomes so after an *origin-preserving* conformal transform, after which we can once again resort to Section 2.3. This is no serious complication, because such conformal transforms don't diminish the accuracy with which Taylor coefficients of a given rank are computed;
(iii) the singularity under investigation is not closest to zero, nor can it be made so under a reasonably simple, origin-preserving conformal transform. We must then take recourse to *origin-changing* conformal transforms, the simplest instances of which are *shifts*. This is the least favourable case, because origin-changing conformal transforms – and be they simple shifts – entail a steep loss of numerical accuracy and demand great attention to the propriety of the truncations being performed.[84]

Fortunately, this third, least favourable situation shall occur but once (in Section 9.8.3, when investigating the arrow $Lo \to Lii$) and even there we will manage the confirm the theoretical prediction with reasonable accuracy (up to 7 places). In all other instances, we shall achieve truly remarkable numerical accuracy, often with up to 50 or 60 exact digits.

9.8.1 From ***Li*** **to** ***Lii*** **and back (*inner* to *inner*)** Since the theory predicts that Li and Lii generate each other under alien differentiation, but that

[84] Indeed, inept truncations can all too easily lead to meaningless results.

neither of them generates Lo nor Lu, we may directly solve the system $\mathbb{S}_{i\,,\,ii}^{\mathbf{n},\mathbf{m}}$:

$$\tilde{\ell i}_{-\frac{1}{2}+n} = 3\,\nu_{1,3}^{\frac{1}{2}-n} \sum_{0\le m<\mathbf{m}} r\tilde{i}\tilde{i}s_{\frac{1}{2}+m} \left(-\frac{1}{2}+n\right)^{-\frac{1}{2}-m} \quad ; \quad \forall n \in]\mathbf{n}-\mathbf{m},\mathbf{n}]$$

with $\mathbf{n}$ equations and $\mathbf{m}$ unknowns $r\tilde{i}\tilde{i}s_{\frac{1}{2}+m}$. Then we may form:

$$r\overset{\wedge}{ii}s\;(\rho) := \sum_{0\le m<\mathbf{m}} r\tilde{i}\tilde{i}s_{\frac{1}{2}+m} \frac{\rho^{-\frac{1}{2}+m}}{\left(-\frac{1}{2}+m\right)!}$$

$$l\widehat{ii}s\;(\nu) := r\overset{\wedge}{ii}s\left(\log\left(1+\frac{\nu}{\nu_{1,3}}\right)\right) = \sum_{0\le m<\mathbf{m}} l\widehat{ii}s_{-\frac{1}{2}+m}\;\nu^{-\frac{1}{2}+m}$$

and check that the ratios $rat_{-\frac{1}{2}+m} := \dfrac{l\widehat{ii}s_{-\frac{1}{2}+m}}{l\widehat{ii}_{-\frac{1}{2}+m}}$ are indeed ~ 1. For instance, with the coefficients $l\widehat{ii}s_{-\frac{1}{2}+m}$ computed from $\mathbb{S}_{i\,,\,ii}^{150,45}$, we already get a high degree of accuracy:

$$|1-rat_{-\frac{1}{2}}| < 10^{-58},\ldots,|1-rat_{\frac{15}{2}}| < 10^{-40},\ldots,|1-rat_{\frac{31}{2}}| < 10^{-24},\ldots$$

This confirms the (equivalent) pairs of resurgence equations

$$\Delta_{\nu_3-\nu_1}\overset{0}{\ell i} = \frac{3}{2\pi}\overset{0}{\ell ii}; \quad \Delta_{\nu_1-\nu_3}\overset{0}{\ell ii} = \frac{3}{2\pi}\overset{0}{\ell i}$$
$$\Delta_{\nu_3-\nu_1}\overset{\Diamond}{\ell i} = \frac{3}{2\pi}\overset{\Diamond}{\ell ii}; \quad \Delta_{\nu_1-\nu_3}\overset{\Diamond}{\ell ii} = \frac{3}{2\pi}\overset{\Diamond}{\ell i}$$

in the ν-plane, which in turn imply

$$\Delta_{\zeta_3-\zeta_1}\overset{\Diamond}{Li} = \frac{3}{2\pi}\overset{\Diamond}{Lii}; \quad \Delta_{\zeta_1-\zeta_3}\overset{\Diamond}{Lii} = \frac{3}{2\pi}\overset{\Diamond}{Li}$$

in the ζ-plane.

9.8.2 From *Lo* to *Li* (*original* to *close-inner*) Since ζ_1 is closest to 0, we solve the system $\mathbb{S}_{o\,,\,i}^{\mathbf{n},\mathbf{m}}$:

$$\overset{\wedge}{Lo}_n = 2\,\zeta_1^{-n} \sum_{0\le m<\mathbf{m}} l\tilde{i}s_{-\frac{1}{2}+m}\; n^{\frac{1}{2}-m}; \quad \forall n \in]\mathbf{n}-\mathbf{m},\mathbf{n}]$$

with $\mathbf{n}$ equations and $\mathbf{m}$ unknowns $\tilde{lis}_{-\frac{1}{2}+m}$. Then we check that the ratios $rat_{-\frac{1}{2}+m} := \frac{\widehat{lis}_{-\frac{1}{2}+m}}{\widehat{li}_{-\frac{1}{2}+m}}$ are indeed ~ 1. For instance, with the coefficients $\widehat{lis}_{-\frac{1}{2}+m}$ computed from $\mathbb{S}^{700,50}_{o,i}$, we get this sort of accuracy:

$$|1 - rat_{-\frac{1}{2}}| < 10^{-54}, \ldots, |1 - rat_{\frac{15}{2}}| < 10^{-29}, \ldots, |1 - rat_{\frac{31}{2}}| < 10^{-6}, \ldots$$

This confirms the resurgence equations $\Delta_{\zeta_1} \overset{\diamond}{Lo} = 2\ \overset{\diamond}{Li}$ in the ζ-plane.

9.8.3 From *Lo* to *Lii* (*original* to *distant-inner*) The singular point ζ_3 being farthest from 0, we first resort to an origin-preserving conformal transform $\zeta \to \xi$:

$$\begin{array}{ll}
h_{\zeta,\xi} : \xi \mapsto \zeta := \zeta_1 - \left(\zeta_1^{1/4} - \xi\right)^4 & \forall\, \xi \\
h_{\xi,\zeta} : \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta_1 - \zeta\right)^{1/4} & \forall\, \zeta \in [0, \zeta_1] \\
h^{+}_{\xi,\zeta} : \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta - \zeta_1\right)^{1/4} e^{-i\pi/4} & \forall\, \zeta \in [\zeta_1, \infty] \\
h^{-}_{\xi,\zeta} : \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta - \zeta_1\right)^{1/4} e^{+i\pi/4} & \forall\, \zeta \in [\zeta_1, \infty]
\end{array}$$

$$\begin{array}{lll}
h_{\xi,\zeta} : \zeta_1 \mapsto \xi_1 = 0.9224\ldots & ;\ |\xi_1| = 0.9224\ldots & \text{(farthest)} \\
h^{\pm}_{\xi,\zeta} : \zeta_2 \mapsto \xi_2^{\pm} = 0.4098 \pm 0.5126\, i \ldots & ;\ |\xi_2^{\pm}| = 0.6563\ldots & \text{(closest)} \\
h^{\pm}_{\xi,\zeta} : \zeta_3 \mapsto \xi_3^{\pm} = 0.2857 \pm 0.6367\, i \ldots & ;\ |\xi_3^{\pm}| = 0.6979\ldots & \text{(middling).}
\end{array}$$

Since the images $\xi_3^{\pm}$ are closer, but not closest, to 0, we must perform an additional shift $\xi \to \tau$:

$$h_{\tau,\xi} : \xi \mapsto \tau := \xi - \tfrac{i}{2} \qquad\qquad h_{\xi,\tau} : \tau \mapsto \xi := \tau + \tfrac{i}{2}$$

$$\begin{array}{lll}
h_{\tau,\xi} : \xi_1 \mapsto \tau_1 = 0.9224 - 0.5000\, i \ldots & |\tau_1| = 1.0492\ldots & \text{(farthest)} \\
h_{\tau,\xi} : \xi_2^{+} \mapsto \tau_2^{+} = 0.4098 + 0.0125\, i \ldots & |\tau_2^{+}| = 0.4100\ldots & \text{(middling)} \\
h_{\tau,\xi} : \xi_3^{+} \mapsto \tau_3^{+} = 0.2857 + 0.1367\, i \ldots & |\tau_3^{+}| = 0.3167\ldots & \text{(closest).}
\end{array}$$

The image τ_3^{+} at last is closest, and we can now go through the usual motions. We form successively:

$$\overset{\wedge}{Lo}_{\#}(\zeta) := \sum_{0<n<\mathbf{n}} \overset{\wedge}{Lo}_n\ \zeta^n \qquad \text{(truncation)}$$

$$\overset{\wedge}{Lo}_{\#\#}(\xi) := \overset{\wedge}{Lo}_{\#}(h_{\zeta,\xi}(\xi)) \qquad \text{(conf. transf.)}$$

$$\overset{\wedge}{Lo}_{\#\#\#}(\xi) := \overset{\wedge}{Lo}_{\#\#}(h_{\xi,\tau}(\tau)) = \sum_{0<n<\mathbf{n}} L_n\ \tau^n + (\ldots) \qquad \text{(simple shift).}$$

We then solve the system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{o\,,ii}$:

$$L_n = 4\,i\,(\tau_3^+)^{-n} \sum_{0\le m<\mathbf{m}} P_{-\frac{1}{2}+m}\ \ n^{\frac{1}{2}-m} \qquad \big(n\ \in]\mathbf{n}-\mathbf{m},\mathbf{n}]\big)$$

with $\mathbf{m}$ equations and $\mathbf{m}$ unknowns $P_{-\frac{1}{2}+m}$.

$$\hat{P}\,(\nu) := \sum_{0\le m<\mathbf{m}} P_{-\frac{1}{2}+m}\ \frac{\nu^{-\frac{3}{2}+m}}{\left(-\frac{3}{2}+m\right)!} + (\ldots)$$

$$\hat{R}\,(\tau) := \hat{P}\left(\log\left(1+\frac{\tau}{\tau_3^+}\right)\right) = \sum_{0\le m<\mathbf{m}} R_{-\frac{3}{2}+m}\ \ \tau^{-\frac{3}{2}+m} + (\ldots)$$

Next, for comparison, we form series that carry the expected singularity Lii successively in the ν, ζ and τ-planes:

$$\overset{\wedge}{\ell ii}\,(\nu) := \sum_{0\le m<\mathbf{m}} \tilde{\ell ii}_{-\frac{1}{2}+m} \frac{\nu^{-\frac{3}{2}+m}}{\left(-\frac{3}{2}+m\right)!} + (\ldots)$$

$$\overset{\wedge}{Lii}\,(\zeta) := \overset{\wedge}{\ell ii}\left(\log\left(1+\frac{\zeta}{\zeta_3}\right)\right)$$

$$\hat{Q}\,(\tau) := \overset{\wedge}{Lii}\,(dh_{\zeta,\tau}(\tau)) \ = \sum_{0\le m<\mathbf{m}} Q_{-\frac{3}{2}+m}\tau^{-\frac{3}{2}+m} + (\ldots).$$

Lastly, we form the ratios $rat_{-\frac{3}{2}+m} := \frac{R_{-\frac{3}{2}+m}}{Q_{-\frac{3}{2}+m}}$ of homologous coefficients P, Q and check that these ratios are ~ 1. With the data derived from the linear system $\mathbb{S}^{800,4}_{o\,,ii}$ and with truncation at order $\mathbf{n}^* = \mathbf{20}$ in the computation of $\overset{\wedge}{Lo}_{\#\#\#}$, we get the following, admittedly poor degree[85] of accuracy:

$$|1 - rat_{-3/2}| < 10^{-7},\ |1 - rat_{-1/2}| < 10^{-3},\ |1 - rat_{+1/2}| < 10^{-2}, \ldots$$

To compound the poor numerical accuracy, the theoretical interpretation is also rather roundabout in this case. By itself, the above results only show that:

$$\Delta^{\pm}_{\zeta_3}\ \overset{\Diamond}{Lo} = \pm 4\,i\ \overset{\Diamond}{Lii} \tag{9.18}$$

[85] This is because of the recourse to the *shift* $\tau := \xi + \frac{i}{2}$ whereas in all the other computations we handled less disruptive *origin-preserving* conformal transforms $\zeta \to \xi$.

with the one-path lateral operators $\Delta_\omega^\pm$ of Section 3 which, unlike the multi-path averages Δ_ω, are *not* alien derivations. To infer from (9.18) the expected resurgence equation:

$$\Delta_{\zeta_3} \overset{\diamond}{Lo} = 0 \overset{\diamond}{Lii} \tag{9.19}$$

we must apply the basic identity (5) of Section 2.3 to $\overset{\diamond}{Lo}$:

$$\left(1 + \sum_{0<\omega} \Delta_\omega^+\right) \overset{\diamond}{Lo} = \left(\exp\left(2\pi i \sum_{0<\omega} \Delta_\omega\right)\right) \overset{\diamond}{Lo} \tag{9.20}$$

and then equate the sole term coming from the left-hand side, namely $\Delta_{\zeta_3}^\pm \overset{\diamond}{Lo}$, with the 4 possible terms coming from the right-hand side, namely:

$$2\pi i\ \Delta_{\zeta_3} \overset{\diamond}{Lo} = \text{unknown} \tag{9.21}$$

$$\frac{(2\pi i)^2}{2} \Delta_{\zeta_3-\zeta_1}\Delta_{\zeta_1} \overset{\diamond}{Lo} = 1\ \overset{\diamond}{Lii} \tag{9.22}$$

$$\frac{(2\pi i)^2}{2} \Delta_{\zeta_3-\zeta_2}\Delta_{\zeta_2} \overset{\diamond}{Lo} = 3\ \overset{\diamond}{Lii} \tag{9.23}$$

$$\frac{(2\pi i)^3}{6} \Delta_{\zeta_3-\zeta_2}\Delta_{\zeta_2-\zeta_1}\Delta_{\zeta_1} \overset{\diamond}{Lo} = 0\ \overset{\diamond}{Lii}\,. \tag{9.24}$$

Equating the terms in the left and right clusters, we find that the sole unknown term (9.21) does indeed vanish, as required by the theory.

9.8.4 From *Lo* **to** *Lu* **(*original* to *outer*)** A single, origin-preserving conformal transform $\zeta \to \xi$ takes the singular point ζ_2 to middling position $\xi_2^\pm$:

$$\begin{aligned}
h_{\zeta,\xi} &: \xi \mapsto \zeta := \zeta_1 - \left(\zeta_1^{1/2} - \xi\right)^2 && \forall \xi \\
h_{\xi,\zeta} &: \zeta \mapsto \xi := \zeta_1^{1/2} - \left(\zeta_1 - \zeta\right)^{1/2} && \forall \zeta \in [0, \zeta_1] \\
h_{\xi,\zeta}^+ &: \zeta \mapsto \xi := \zeta_1^{1/2} + i\left(\zeta - \zeta_1\right)^{1/2} && \forall \zeta \in [\zeta_1, \infty] \\
h_{\xi,\zeta}^- &: \zeta \mapsto \xi := \zeta_1^{1/2} - i\left(\zeta - \zeta_1\right)^{1/2} && \forall \zeta \in [\zeta_1, \infty]
\end{aligned}$$

$$\begin{aligned}
h_{\xi,\zeta} &: \zeta_1 \mapsto \xi_1 = 0.8508\ldots && ;\ |\xi_1| = 0.8508\ldots\ \text{(closest)} \\
h_{\xi,\zeta}^\pm &: \zeta_2 \mapsto \xi_2^\pm = 0.8508 \pm 0.5254\, i \ldots && ;\ |\xi_2^\pm| = 1.0000\ldots\ \text{(middling)} \\
h_{\xi,\zeta}^\pm &: \zeta_3 \mapsto \xi_3^\pm = 0.8508 \pm 0.8108\, i \ldots && ;\ |\xi_3^\pm| = 1.7573\ldots\ \text{(farthest).}
\end{aligned}$$

Then we form:

$$\overset{\wedge}{Lo}_{\#}\ (\zeta) := \sum_{0<n<\mathbf{n}} \overset{\wedge}{Lo}_n\ \ \zeta^n \qquad \text{(truncation)}$$

$$\overset{\wedge}{Lo}_{\#\#}\ (\xi) := \overset{\wedge}{Lo}_{\#}\,(h_{\zeta,\xi}(\xi)) \qquad \text{(conf. transf.)}$$

$$\overset{\wedge}{Lo}_{\#\#\#}\ (\xi) := \overset{\wedge}{Lo}_{\#\#}\,(\xi)\ (\xi_1 - \xi)^3 = \sum_{0<n<\mathbf{n}} L_n\ \xi^n + (\dots) \quad \text{(sing. remov.)}.$$

Since $\zeta_1 - \zeta\ = (\xi_1 - \xi)^2$, all the semi-integral powers $(\zeta_1 - \zeta)^{n/2}$ present in $\overset{\wedge}{Lo}_{\#}\ (\zeta)$ at $\zeta\ \sim \zeta_2$ vanish from $\overset{\wedge}{Lo}_{\#\#}\ (\xi)$, except for the first two terms:

$$C_{-3}\ (\xi_1 - \xi)^{-3} + C_{-1}\ (\xi_1 - \xi)^{-1}$$

but even these two vanish from $\overset{\wedge}{Lo}_{\#\#\#}\ (\xi)$ due to multiplication by $(\xi_1 - \xi)^3$. So the points $\xi_2^{\pm}$ now carry the closest singularities of $\overset{\wedge}{Lo}_{\#\#\#}\ (\xi)$, and we can apply the usual Taylor coefficient asymptotics.

For comparison with the expected singularity $\overset{\wedge}{Lu}$, we construct a new triplet $\{\overset{\wedge}{Ro}_{\#}, \overset{\wedge}{Ro}_{\#\#}, \overset{\wedge}{Ro}_{\#\#\#}\}$, but with a more severe truncation ($\mathbf{n}^* \prec \mathbf{n}$) and with coefficients $\overset{\wedge}{Lo}_n$ replaced by the $\overset{\wedge}{Ro}_n$ defined as follows:

$$\overset{\wedge}{Ro}_n := \frac{1}{n} SP^F\left(\frac{1}{n}\right) \qquad \text{with} \qquad SP^F(x) := \sum_{1\le m\le \mathbf{n}^*}\ \prod_{1\le k\le m} F(k\,x)$$

$$\overset{\wedge}{Ro}_{\#}\ (\zeta) := \sum_{0<n<\mathbf{n}^*} \overset{\wedge}{Ro}_n\ \ \zeta^n \qquad \text{(truncation)}$$

$$\overset{\wedge}{Ro}_{\#\#}\ (\xi) := \overset{\wedge}{Ro}_{\#}\,(h_{\zeta,\xi}(\xi)) \qquad \text{(conf. transf.)}$$

$$\overset{\wedge}{Ro}_{\#\#\#}\ (\xi) := \overset{\wedge}{Ro}_{\#\#}\,(\xi)(\xi_1 - \xi)^3 = \sum_{0<n<\mathbf{n}} R_n \xi^n + (\dots) \quad \text{(sing. remov.)}.$$

Then, we solve the two parallel systems $\overline{\mathbb{S}}^{\mathbf{n},\mathbf{m}}_{o\,,\,u}$ and $\underline{\mathbb{S}}^{\mathbf{n},\mathbf{m}}_{o\,,\,u}$:

$$L_n = \sum_{\epsilon=\pm} (\xi_2^{\epsilon})^{-n} \sum_{1\le m\le \mathbf{m}} L_m^{\epsilon}\ n^{-k} \qquad (n\ \in\ ]\mathbf{n} - 2\,\mathbf{m}, \mathbf{n}])$$

$$R_n = \sum_{\epsilon=\pm} (\xi_2^{\epsilon})^{-n} \sum_{1\le m\le \mathbf{m}} R_m^{\epsilon}\ n^{-k} \qquad (n\ \in\ ]\mathbf{n} - 2\,\mathbf{m}, \mathbf{n}])$$

each with $2\,\mathbf{m}$ equations and $2\,\mathbf{m}$ unknowns, L_m^ϵ or R_m^ϵ respectively. We then check that the ratios $rat_n^\epsilon := \frac{L_m^\epsilon}{R_m^\epsilon}$ are ~ 1. With the data obtained from the systems $\overline{\mathbb{S}}_{o,u}^{495,7}$ and $\underline{\mathbb{S}}_{o,u}^{495,7}$ and with truncation at order $\mathbf{n}^* = \mathbf{30}$ in the $\hat{R}\, o$ triplet, we get the following degree of accuracy:

$$|1 - rat_1^{\pm}| < 10^{-17}, \ldots, |1 - rat_3^{\pm}| < 10^{-13}, \ldots, |1 - rat_6^{\pm}| < 10^{-10}, \ldots,$$

The immediate implication is $\Delta_{\zeta_2}^{+} \overset{\diamond}{Lo} = 2\pi i\ \overset{\diamond}{Lu}$. To translate this into a statement about $\Delta_{\zeta_2} \overset{\diamond}{Lo}$, the argument is the same as in Section 9.8.3, only much simpler. Indeed, the only term coming from the left-hand side of (9.20) is now $\Delta_{\zeta_2}^{+} \overset{\diamond}{Lo}$ and the only two possible terms coming from the right-hand side are:

$$2\pi i\ \Delta_{\zeta_2} \overset{\diamond}{Lo} = \text{unknown} \quad \text{and} \quad \frac{(2\pi i)^2}{2} \Delta_{\zeta_2-\zeta_1} \Delta_1 \overset{\diamond}{Lo} = 0. \qquad (9.25)$$

Equating both sides, we find $\Delta_{\zeta_2} \overset{\diamond}{Lo} = \overset{\diamond}{Lu}$, as required by the theory.

9.8.5 From *Lu* to *Li* and *Lii* (*outer* to *inner*) The singular points under investigation being closest, the investigation is straightforward. We form the linear system $\mathbb{S}_{u,i/ii}^{\mathbf{n},\mathbf{m}}$:

$$\overset{\wedge}{\ell u}_n = 2\,\left(\nu_3^{-n} + (-\nu_3)^{-n}\right) \sum_{0 \le m < 2\,\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\, n^{\frac{1}{2}-m} \qquad (n\ \in\ ]\mathbf{n}-\mathbf{2m}, \mathbf{n}])$$

with $\mathbf{m}$ effective equations (for even values of n) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. We then form:

$$\overset{\wedge}{riis}\ (\xi) := \sum_{0 \le m < \mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m} \frac{\xi^{-\frac{3}{2}+m}}{\left(-\frac{3}{2}+m\right)!}$$

$$\overset{\wedge}{liis}\ (\xi) := \overset{\wedge}{riis}\left(\log\left(1 + \frac{\nu}{\nu_3}\right)\right) = \sum_{0 \le m < \mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m}\ \nu^{-\frac{3}{2}+m}$$

and check that the ratios $rat_{-\frac{3}{2}+m} := \dfrac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}_{u,i/ii}^{300,40}$, we get this high degree of accuracy:

$$|1-rat^{\pm}_{-3/2}| < 10^{-81},\ \ldots\ ,|1-rat^{\pm}_{17/2}| < 10^{-39},\ \ldots\ ,$$

$$|1-rat^{\pm}_{37/2}| < 10^{-21},\ \ldots\ ,|1-rat^{\pm}_{57/2}| < 10^{-10},\ \ldots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, namely:

$$\Delta_{\zeta_3-\zeta_2}\overset{\Diamond}{Lu} = \frac{2}{2\pi}\overset{\Diamond}{Lii};\quad \Delta_{\zeta_1-\zeta_2}\overset{\Diamond}{Lu} = \frac{2}{2\pi}\overset{\Diamond}{Li}.$$

9.8.6 From *Loo* **to** *Li* **(*original* to *close-inner*)** We proceed exactly as in Section 9.8.2. We form the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{oo\,,\,i}$:

$$\overset{\wedge}{Loo}_n = \zeta_1^{-n}\sum_{0\leq m<\mathbf{m}} \tilde{lis}_{-\frac{1}{2}+m}\, n^{\frac{1}{2}-m} \qquad (n\ \in\ ]\mathbf{n}-2\mathbf{m},\mathbf{n}])$$

with $\mathbf{m}$ equations and $\mathbf{m}$ unknowns $\tilde{lis}_{-\frac{1}{2}+m}$. We then check that the ratios $rat_{-\frac{3}{2}+m} := \dfrac{\overset{\wedge}{lis}_{-\frac{3}{2}+m}}{\overset{\wedge}{li}_{-\frac{3}{2}+m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}^{800,30}_{oo\,,\,i}$, we get this degree of accuracy:

$$|1-rat^{\pm}_{-1/2}|<10^{-51},\ldots,|1-rat^{\pm}_{19/2}|<10^{-22},\ldots,|1-rat^{\pm}_{39/2}|<10^{-7},\ldots$$

This confirms the expected resurgence equations in the ζ-plane:

$$\Delta_{\zeta_1}\overset{\Diamond}{Lo} = 2\,\overset{\Diamond}{Li};\quad \Delta_{\underline{\zeta_1}}\overset{\Diamond}{Lo} = 0\ \underline{\overset{\Diamond}{Li}}.$$

An alternative method would to check that $\overset{\wedge}{Lo}\,(\zeta) - \overset{\wedge}{Loo}\,(\zeta)$ has radius of convergence 1, which means that $\overset{\wedge}{Lo}$ and $\overset{\wedge}{Loo}$ have the same singularity at ζ_1, namely $\overset{\Diamond}{Li}$: see Section 9.8.2. With that method, too, the numerical confirmation is excellent.

9.8.7 From *Loo* **to** *Lii* **(*original* to *distant-inner*)** The verication hasn't been done yet. The theory, however, predicts a vanishing alien derivative $\Delta_{\zeta_3}(\overset{\Diamond}{Loo}) = 0$ just as with $\overset{\Diamond}{Lo}$. Therefore, the upper/lower lateral singularity seen at ζ_3 when continuing $\overset{\Diamond}{Loo}\,(\zeta)$ should be $\pm 4\,i\,\overset{\Diamond}{Lii}$, just as was the case with the lateral continuations of $\overset{\Diamond}{Lo}\,(\zeta)$.

9.8.8 From *Loo* **to** *Luu* (*original* **to** *outer*) We form the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{oo\,,\,uu}$:

$$\overset{\wedge}{Loo}_n - \overset{\wedge}{Lo}_n = -\zeta_2^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{lu}_{1+2m}\; n^{-1-2m}$$

$$+ \zeta_2^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{luus}_{2m}\;\; n^{-2m}$$

$$- 2;\, (-\zeta_2)^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{luus}_{2m}\;\; n^{-2m}$$

with $\mathbf{m}$ equations[86] and $\mathbf{m}$ unknowns $\tilde{luus}_{2m}$.[87] We then check that the ratios $rat_{2m} := \frac{\overset{\wedge}{luus}_{2m}}{\overset{\wedge}{luu}_{2m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}^{600,30}_{oo\,,\,uu}$, we get this level of accuracy:

$$|1 - rat_2| < 10^{-48}, \ldots, |1 - rat_{12}| < 10^{-28}, \ldots, |1 - rat_{24}| < 10^{-15}, \ldots$$

This directly confirms the expected resurgence equations:

$$\Delta_{\zeta_2} \overset{\Diamond}{Loo} = \overset{\Diamond}{Luu}; \qquad \Delta_{\underline{\zeta_2}} \overset{\Diamond}{Loo} = -2\, \underline{\overset{\Diamond}{Luu}}$$

9.8.9 From *Luu* **to** *Li* **and** *Lii* (*outer* **to** *inner*) We proceed exactly as in Section 9.8.8. We solve the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{uu\,,\,i/ii}$:

$$\overset{\wedge}{luu}_n = \frac{2}{2\pi}\, (\nu_3^{-n} - (-\nu_3)^{-n}) \sum_{0\le m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\; n^{\frac{1}{2}-m} \qquad (n \in]\mathbf{n}\text{–}\mathbf{2m},\ \mathbf{n}])$$

with $\mathbf{m}$ effective equations (for n odd) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. Then we form:

$$\overset{\wedge}{riis}\;(\xi) := \sum_{0\le m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m} \frac{\xi^{-\frac{3}{2}+m}}{\left(-\frac{3}{2}+m\right)!} + (\ldots)$$

$$\overset{\wedge}{liis}\;(\xi) := \overset{\wedge}{riis}\left(\log\left(1 + \frac{\nu}{\nu_3}\right)\right) =: \sum_{0\le m<\mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m}\;\; \nu^{-\frac{3}{2}+m} + (\ldots)$$

[86] With n ranging through the interval $]\mathbf{n}-\mathbf{2m}, \mathbf{n}]$.

[87] The coefficients $\tilde{lu}_{1+2m}$ are already known, from Section 9.8.4.

and check that the ratios $rat_{-\frac{3}{2}+m} := \dfrac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ of homologous coefficients are indeed ~ 1. For the data corresponding to $\mathbb{S}^{300,40}_{uu\,,\,i/ii}$, we find this excellent level of accuracy:

$$|1-rat_{-\frac{3}{2}}|<10^{-74},\ldots,|1-rat_{\frac{21}{2}}|<10^{-37},\ldots,|1-rat_{\frac{45}{2}}|<10^{-18},\ldots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, namely:

$$\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Luu} = \frac{2}{2\pi}\ \overset{\diamond}{Lii};\quad \Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Luu} = -\frac{2}{2\pi}\ \overset{\diamond}{Li}$$

and also, by mirror symmetry:

$$\Delta_{\underline{\zeta_3}-\underline{\zeta_2}}\overset{\diamond}{\underline{Luu}} = \frac{2}{2\pi}\ \overset{\diamond}{\underline{Lii}};\quad \Delta_{\underline{\zeta_1}-\underline{\zeta_2}}\overset{\diamond}{\underline{Luu}} = -\frac{2}{2\pi}\ \overset{\diamond}{\underline{Li}}$$

9.8.10 From *Le* to *Li* and *Lii* (*exceptional* to *inner*) As in the preceding subsection, we solve the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{e\,,\,i/ii}$:

$$\overset{\wedge}{le}_n = \frac{2}{2\pi}\,(\nu_3^{-n}-(-\nu_3)^{-n})\sum_{0\le m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\; n^{\frac{1}{2}-m} \qquad (n\in]\mathbf{n}-\mathbf{2m},\ \mathbf{n}])$$

with $\mathbf{m}$ effective equations (for n odd) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. Then we form:

$$\overset{\wedge}{riis}\,(\xi) := \sum_{0\le m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\frac{\xi^{-\frac{3}{2}+m}}{\left(-\frac{3}{2}+m\right)!} + (\ldots)$$

$$\overset{\wedge}{liis}\,(\xi) := \overset{\wedge}{riis}\left(\log\left(1+\frac{\nu}{\nu_3}\right)\right) = \sum_{0\le m<\mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m}\ \ \nu^{-\frac{3}{2}+m} + (\ldots)$$

and check that the ratios $rat_{-\frac{3}{2}+m} := \dfrac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ of homologous coefficients are indeed ~ 1. For the data corresponding to $\mathbb{S}^{300,40}_{e\,,\,i/ii}$, we find this excellent level of accuracy:

$$|1-rat_{-\frac{3}{2}}|<10^{-73},\ldots,|1-rat_{\frac{21}{2}}|<10^{-38},\ldots,|1-rat_{\frac{45}{2}}|<10^{-17},\ldots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, to wit:

$$\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Le} = \frac{2}{2\pi}\ \overset{\diamond}{Lii};\quad \Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Le} = -\frac{2}{2\pi}\ \overset{\diamond}{Li}$$

An alternative method is to check that $\overset{\wedge}{\ell}\ \cancel{e}(\nu) - \overset{\wedge}{\ell}\ u\cancel{u}(\nu)$ has a radius of convergence larger than $|\nu_1| = |\nu_3|$, which implies that $\overset{\wedge}{\ell}\ e$ and $\overset{\wedge}{\ell}\ uu$ have the same singularity at ν_1 and ν_3, namely $\frac{2}{2\pi}\ \overset{\Diamond}{Li}$ and $\frac{2}{2\pi}\ \overset{\Diamond}{Lii}$: see Section 9.8.9. Here too, the numerical accuracy is excellent.

10 General tables

10.1 Main formulas

10.1.1 Functional transforms

standard case	$\beta(\tau) := \dfrac{1}{e^{\tau/2)} - e^{-\tau/2}}$	$\beta^\dagger(\tau) := \dfrac{1}{e^{\tau/2)} - e^{-\tau/2}} - \dfrac{1}{\tau}$
free β case	$\beta(\tau) := \tau^{-1} + \sum_{1\le k} \beta_k\, \tau^k$	$\beta^\dagger(\tau) := \sum_{1\le k} \beta_k\, \tau^k$

. . . .

mir-transform: $\cancel{g} := 1/g \mapsto \hbar := 1/h$ with

$$\frac{1}{\hbar(\nu)} = \left[\frac{1}{\cancel{g}(\nu)} \exp\left(-\beta^\dagger\big(I\,\cancel{g}(\nu)\partial_\nu\big)\,\cancel{g}(\nu)\right)\right]_{I=\partial_\nu^{-1}} \tag{10.1}$$

. . . .

nir-transform: $f \mapsto h$ with

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\int_0^{+\infty} \exp^{\#}\Big(-\beta(\partial_\tau)\, f\left(\frac{\tau}{n}\right)\Big)\, d\tau \tag{10.2}$$

. . . .

nir-translocation: $f \mapsto \nabla h := (nir - e^{-\eta\partial_\nu} nir\, e^{\epsilon\partial_x})(h)$ with

$$\nabla h(\epsilon, \nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\int_0^{\epsilon\, n} \exp_{\#}\Big(-\beta(\partial_\tau)\, f\left(\frac{\tau}{n}\right)\Big)\, d\tau \tag{10.3}$$

. . . .

nur-transform: $f \mapsto h$ with

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\sum_{\tau\in\frac{1}{2}+\mathbb{N}} \exp^{\#}\Big(-\beta(\partial_\tau)\, f\left(\frac{\tau}{n}\right)\Big)\, d\tau \tag{10.4}$$

. . . .

nur in terms of *nir* :

$$\mathrm{nur}(f) = \sum_{k\in\mathbb{Z}} (-1)^k\, \mathrm{nir}(k\, 2\pi i + f) \tag{10.5}$$

. . . .

For the interpretation of $\exp^{\#}$, $\exp_{\#}$ see Section 4.3.

10.1.2 SP coefficients and SP series

Basic data: $F = \exp(-f), \quad \eta_F := \int_0^1 f(x)dx, \quad \omega_F = e^{-\eta_F}$

asymptotic series	funct. germs
$\tilde{Ig}_F(n) = \exp\left(-\frac{1}{2}f(0) + \sum_{1\leq s \text{odd}} \frac{\mathfrak{b}_s}{n^s} f^{(s)}(0)\right)$	$Ig_F(n)$ ingress factor
$\tilde{Eg}_F(n) = \exp\left(-\frac{1}{2}f(1) - \sum_{1\leq s \text{odd}} \frac{\mathfrak{b}_s}{n^s} f^{(s)}(1)\right)$	$Eg_F(n)$ egress factor

....

"raw″	"cleansed″
$P_F(n) := \prod_{0\leq k\leq n} F\left(\frac{k}{n}\right)$	$P_F^{\#}(n) := (\omega_F)^n = \frac{P_F(n)}{Ig_F(n)\,Eg_F(n)}$
$J_F(n) := \sum_{0\leq m<n} \prod_{0\leq k\leq m} F\left(\frac{k}{n}\right)$	$J_F^{\#}(n) := J_F(n)/Ig_F(n)$
$j_F(\zeta) := \sum_{0\leq n} J_F(n)\, \zeta^n$	$j_F^{\#}(\zeta) := \sum_{0\leq n} J_F^{\#}(n)\, \zeta^n.$

10.1.3 Parity relations

$$F^{\vDash}(x) := 1/F(1-x) \qquad \Longrightarrow$$

$$1 = \tilde{Ig}_F(n)\, \tilde{Eg}_{F^{\vDash}}(n) = \tilde{Ig}_{F^{\vDash}}(n)\, \tilde{Eg}_F(n)$$

$$J_{F^{\vDash}}(n) = J_F(n)/P_F(n) \quad \text{and} \quad J^{\#}_{F^{\vDash}}(n) = J^{\#}_F(n)/P^{\#}_F(n)$$

$$j_{F^{\vDash}}(\zeta) \neq j_F(\zeta/\omega_F) \quad \text{but} \quad j^{\#}_{F^{\vDash}}(\zeta) = j^{\#}_F(\zeta/\omega_F)$$

....

$$F^{\perp}(x) := 1/F(-x), \quad f^{\perp}(x) := -f(-x) \qquad \Longrightarrow$$

$$\text{nur}(f^{\perp})(\nu) = -\text{nur}(f)(\nu) \qquad (\text{tangency } \kappa = 0)$$

$$\text{nir}(f^{\perp})(\nu) = -\text{nir}(f)(\nu) \qquad (\text{tangency } \kappa = 0)$$

$$\text{nir}(f^{\perp}) \text{ and } \text{nir}(f) \text{ unrelated} \qquad (\text{tangency } \kappa \text{ even} \geq 2)$$

$$\text{nir}(f^{\perp})(\nu) = -\text{nur}(f)(\epsilon_\kappa \nu) \text{ with } \epsilon_\kappa^{\frac{1}{\kappa+1}} = -1 \qquad (\text{tangency } \kappa \text{ odd} \geq 1)$$

$$\Rightarrow h^{\perp}_{\frac{k}{\kappa+1}} = (-1)^{k-1} h_{\frac{k}{\kappa+1}} \text{ with : } (f, f^{\perp}) \overset{\text{nir}}{\mapsto} (h, h^{\perp}) \quad (\text{tangency } \kappa \text{ odd} \geq 1).$$

....

10.2 The *Mir* mould

10.2.1 Layered form

length: $r = 1$, **order**: $d = 1$, **factor**: $c_{1,1} = 1/24$

$$\mathrm{Mir}[0,1] = 1\ c_{3,2} = 1/24$$

length: $r = 3$, **order**: $d = 2$, **factor**: $c_{3,2} = 1/1152$

$$\mathrm{Mir}[1,2,0,0] = 1\ c_{3,2} = 1/1152$$

length: $r = 3$, **order**: $d = 3$, **factor**: $c_{3,3} = 7/5760$

$$\begin{aligned}
\mathrm{Mir}[0,3,0,0] &= -1\ c_{3,3} = -7/5760\\
\mathrm{Mir}[1,1,1,0] &= -4\ c_{3,3} = -7/1440\\
\mathrm{Mir}[2,0,0,1] &= -1\ c_{3,3} = -7/5760
\end{aligned}$$

length: $r = 5$, **order**: $d = 3$, **factor**: $c_{5,3} = 1/82944$

$$\mathrm{Mir}[2,3,0,0,0,0] = 1\ c_{5,3} = 1/82944$$

length: $r = 5$, **order**: $d = 4$, **factor**: $c_{5,4} = 7/138240$

$$\begin{aligned}
\mathrm{Mir}[1,4,0,0,0,0] &= -1\ c_{5,4} = -7/138240\\
\mathrm{Mir}[2,2,1,0,0,0] &= -4\ c_{5,4} = -7/34560\\
\mathrm{Mir}[3,1,0,1,0,0] &= -1\ c_{5,4} = -7/138240
\end{aligned}$$

length: $r = 5$, **order**: $d = 5$, **factor**: $c_{5,5} = 31/967680$

$$\begin{aligned}
\mathrm{Mir}[0,5,0,0,0,0] &= 1\ c_{5,5} = 31/967680\\
\mathrm{Mir}[1,3,1,0,0,0] &= 26\ c_{5,5} = 403/483840\\
\mathrm{Mir}[2,1,2,0,0,0] &= 34\ c_{5,5} = 527/483840\\
\mathrm{Mir}[2,2,0,1,0,0] &= 32\ c_{5,5} = 31/30240\\
\mathrm{Mir}[3,0,1,1,0,0] &= 15\ c_{5,5} = 31/64512\\
\mathrm{Mir}[3,1,0,0,1,0] &= 11\ c_{5,5} = 341/967680\\
\mathrm{Mir}[4,0,0,0,0,1] &= 1\ c_{5,5} = 31/967680.
\end{aligned}$$

10.2.2 Compact form

length: $r = 1$, **gcd**: $d_3 = 24$

$$^\star\mathrm{Mir}[1] = 1/d_1 = 1/24$$

length: $r = 3$, **gcd**: $d_3 = 5760$

$$\begin{aligned}
^\star\mathrm{Mir}[1,2,0] &= -2/d_3 = -1/2880\\
^\star\mathrm{Mir}[2,0,1] &= -7/d_3 = -7/5760
\end{aligned}$$

length: $r = 5$, **gcd**: $d_5 = 2903040$

$$\begin{aligned}
{}^\star\mathrm{Mir}[1,4,0,0,0] &= 16/d_5 = 1/181440\\
{}^\star\mathrm{Mir}[2,2,1,0,0] &= 540/d_5 = 1/5376\\
{}^\star\mathrm{Mir}[3,0,2,0,0] &= 372/d_5 = 31/241920\\
{}^\star\mathrm{Mir}[3,1,0,1,0] &= 504/d_5 = 1/5760\\
{}^\star\mathrm{Mir}[4,0,0,0,1] &= 93/d_5 = 31/967680
\end{aligned}$$

length: $r = 7$, **gcd**: $d_7 = 1393459200$

$$\begin{aligned}
{}^\star\mathrm{Mir}[1,6,0,0,0,0,0] &= -144/d_7 = -1/9676800\\
{}^\star\mathrm{Mir}[2,4,1,0,0,0,0] &= -28824/d_7 = -1201/58060800\\
{}^\star\mathrm{Mir}[3,2,2,0,0,0,0] &= -141576/d_7 = -5899/58060800\\
{}^\star\mathrm{Mir}[4,0,3,0,0,0,0] &= -38862/d_7 = -2159/77414400\\
{}^\star\mathrm{Mir}[3,3,0,1,0,0,0] &= -88928/d_7 = -397/6220800\\
{}^\star\mathrm{Mir}[4,1,1,1,0,0,0] &= -186264/d_7 = -2587/19353600\\
{}^\star\mathrm{Mir}[5,0,0,2,0,0,0] &= -16116/d_7 = -1343/116121600\\
{}^\star\mathrm{Mir}[4,2,0,0,1,0,0] &= -67878/d_7 = -419/8601600\\
{}^\star\mathrm{Mir}[5,0,1,0,1,0,0] &= -29718/d_7 = -1651/77414400\\
{}^\star\mathrm{Mir}[5,1,0,0,0,1,0] &= -16428/d_7 = -1369/116121600\\
{}^\star\mathrm{Mir}[6,0,0,0,0,0,1] &= -1143/d_7 = -127/154828800
\end{aligned}$$

length: $r = 9$, **gcd**: $d_9 = 367873228800$

$$\begin{aligned}
{}^\star\mathrm{Mir}[1,8,0,0,0,0,0,0,0] &= 768/d_9 = 1/479001600\\
{}^\star\mathrm{Mir}[2,6,1,0,0,0,0,0,0] &= 789504/d_9 = 257/119750400\\
{}^\star\mathrm{Mir}[3,4,2,0,0,0,0,0,0] &= 13702656/d_9 = 811/21772800\\
{}^\star\mathrm{Mir}[4,2,3,0,0,0,0,0,0] &= 26034672/d_9 = 542389/7664025600\\
{}^\star\mathrm{Mir}[5,0,4,0,0,0,0,0,0] &= 3801840/d_9 = 2263/218972160\\
{}^\star\mathrm{Mir}[3,5,0,1,0,0,0,0,0] &= 6324224/d_9 = 193/11226600\\
{}^\star\mathrm{Mir}[4,3,1,1,0,0,0,0,0] &= 52597760/d_9 = 10273/71850240\\
{}^\star\mathrm{Mir}[5,1,2,1,0,0,0,0,0] &= 40989024/d_9 = 47441/425779200\\
{}^\star\mathrm{Mir}[5,2,0,2,0,0,0,0,0] &= 18164736/d_9 = 73/1478400\\
{}^\star\mathrm{Mir}[6,0,1,2,0,0,0,0,0] &= 6350064/d_9 = 18899/1094860800\\
{}^\star\mathrm{Mir}[4,4,0,0,1,0,0,0,0] &= 11628928/d_9 = 90851/2874009600\\
{}^\star\mathrm{Mir}[5,2,1,0,1,0,0,0,0] &= 33372912/d_9 = 695269/7664025600\\
{}^\star\mathrm{Mir}[6,0,2,0,1,0,0,0,0] &= 5886720/d_9 = 73/4561920\\
{}^\star\mathrm{Mir}[6,1,0,1,1,0,0,0,0] &= 9462768/d_9 = 28163/1094860800\\
{}^\star\mathrm{Mir}[7,0,0,0,2,0,0,0,0] &= 429240/d_9 = 511/437944320\\
{}^\star\mathrm{Mir}[5,3,0,0,0,1,0,0,0] &= 7436800/d_9 = 83/4105728\\
{}^\star\mathrm{Mir}[6,1,1,0,0,1,0,0,0] &= 7391376/d_9 = 51329/2554675200\\
{}^\star\mathrm{Mir}[7,0,0,1,0,1,0,0,0] &= 736848/d_9 = 731/364953600\\
{}^\star\mathrm{Mir}[6,2,0,0,0,0,1,0,0] &= 1941144/d_9 = 80881/15328051200\\
{}^\star\mathrm{Mir}[7,0,1,0,0,0,1,0,0] &= 490560/d_9 = 73/54743040\\
{}^\star\mathrm{Mir}[7,1,0,0,0,0,0,1,0] &= 209712/d_9 = 4369/7664025600\\
{}^\star\mathrm{Mir}[8,0,0,0,0,0,0,0,1] &= 7665/d_9 = 73/3503554560\ .
\end{aligned}$$

10.3 The mir transform: from $\not{g}$ to $\hbar$

10.3.1 Tangency 0, ramification 1

Recall that $\not{g} = 1/g$ and $\hbar = 1/h$.

$$
\begin{aligned}
h_1 - g_1 &= \tfrac{1}{24}\not{g}_1 \\
h_2 - g_2 &= \tfrac{1}{24}\not{g}_2 + \tfrac{1}{2304}\not{g}_0\not{g}_1^2 \\
h_3 - g_3 &= \tfrac{1}{24}\not{g}_3 - \tfrac{1}{17280}\not{g}_1^3 - \tfrac{1}{960}\not{g}_0\not{g}_1\not{g}_2 - \tfrac{7}{5760}\not{g}_0^2\not{g}_3 + \tfrac{1}{497664}\not{g}_0^2\not{g}_1^3 \\
h_4 - g_4 &= \tfrac{1}{24}\not{g}_4 - \tfrac{1}{1920}\not{g}_0\not{g}_2^2 - \tfrac{1}{2880}\not{g}_1^2\not{g}_2 - \tfrac{1}{720}\not{g}_0\not{g}_1\not{g}_3 - \tfrac{7}{5760}\not{g}_0^2\not{g}_4 \\
&\quad - \tfrac{23}{1658880}\not{g}_0^2\not{g}_1^2\not{g}_2 - \tfrac{11}{9953280}\not{g}_0\not{g}_1^4 - \tfrac{7}{552960}\not{g}_0^3\not{g}_1\not{g}_3 \\
&\quad + \tfrac{1}{191102976}\not{g}_0^3\not{g}_1^4 \\
h_5 - g_5 &= \tfrac{1}{24}\not{g}_5 - \tfrac{11}{28800}\not{g}_1\not{g}_2^2 - \tfrac{23}{14400}\not{g}_0\not{g}_1\not{g}_4 - \tfrac{7}{14400}\not{g}_1^2\not{g}_3 \\
&\quad - \tfrac{17}{14400}\not{g}_0\not{g}_2\not{g}_3 - \tfrac{7}{5760}\not{g}_0^2\not{g}_5 + \tfrac{143}{21772800}\not{g}_0\not{g}_1^3\not{g}_2 \\
&\quad + \tfrac{61}{2419200}\not{g}_0^2\not{g}_1\not{g}_2^2 + \tfrac{73}{1209600}\not{g}_0^3\not{g}_1\not{g}_4 + \tfrac{19}{537600}\not{g}_0^2\not{g}_1^2\not{g}_3 \\
&\quad + \tfrac{13}{302400}\not{g}_0^3\not{g}_2\not{g}_3 + \tfrac{31}{967680}\not{g}_0^4\not{g}_5 + \tfrac{1}{21772800}\not{g}_1^5 \\
&\quad - \tfrac{37}{597196800}\not{g}_0^3\not{g}_1^3\not{g}_2 - \tfrac{1}{176947200}\not{g}_0^2\not{g}_1^5 \\
&\quad - \tfrac{7}{132710400}\not{g}_0^4\not{g}_1^2\not{g}_3 + \tfrac{1}{114661785600}\not{g}_0^4\not{g}_1^5 \\
h_6 - g_6 &= +\tfrac{1}{24}\not{g}_6 - \tfrac{13}{14400}\not{g}_1\not{g}_2\not{g}_3 - \tfrac{17}{28800}\not{g}_1^2\not{g}_4 - \tfrac{19}{14400}\not{g}_0\not{g}_2\not{g}_4 \\
&\quad - \tfrac{1}{576}\not{g}_0\not{g}_1\not{g}_5 - \tfrac{7}{5760}\not{g}_0^2\not{g}_6 - \tfrac{11}{86400}\not{g}_2^3 - \tfrac{17}{28800}\not{g}_0\not{g}_3^2 \\
&\quad + \tfrac{17}{241920}\not{g}_0^2\not{g}_1\not{g}_2\not{g}_3 + \tfrac{163}{10886400}\not{g}_0\not{g}_1^2\not{g}_2^2 + \tfrac{41}{2721600}\not{g}_0\not{g}_1^3\not{g}_3 \\
&\quad + \tfrac{13}{241920}\not{g}_0^2\not{g}_1^2\not{g}_4 + \tfrac{59}{1209600}\not{g}_0^3\not{g}_2\not{g}_4 + \tfrac{17}{14515200}\not{g}_2\not{g}_1^4 \\
&\quad + \tfrac{13}{181440}\not{g}_0^3\not{g}_1\not{g}_5 + \tfrac{31}{967680}\not{g}_0^4\not{g}_6 + \tfrac{61}{7257600}\not{g}_0^2\not{g}_2^3 \\
&\quad + \tfrac{13}{604800}\not{g}_0^3\not{g}_3{}^2 + \tfrac{461}{1161216000}\not{g}_0^4\not{g}_1\not{g}_2\not{g}_3 + \tfrac{5347}{20901888000}\not{g}_0^3\not{g}_1^2\not{g}_2^2 \\
&\quad + \tfrac{25259}{83607552000}\not{g}_0^3\not{g}_1^3\not{g}_3 + \tfrac{211}{464486400}\not{g}_0^4\not{g}_1^2\not{g}_4 \\
&\quad + \tfrac{12109}{167215104000}\not{g}_0^2\not{g}_1^4\not{g}_2 + \tfrac{31}{139345920}\not{g}_0^5\not{g}_1\not{g}_5 + \tfrac{37}{37158912000}\not{g}_0\not{g}_1^6 \\
&\quad + \tfrac{49}{1327104000}\not{g}_0{}^5\not{g}_3^2 - \tfrac{17}{114661785600}\not{g}_0{}^4\not{g}_1^4\not{g}_2 - \tfrac{1}{68797071360}\not{g}_0^3\not{g}_1^6 \\
&\quad - \tfrac{7}{57330892800}\not{g}_0^5\not{g}_1^3\not{g}_3 + \tfrac{1}{99067782758400}\not{g}_0^5\not{g}_1^6.
\end{aligned}
$$

10.3.2 Tangency 1, ramification 2

$$
\begin{aligned}
h_{1/2} - g_{1/2} &= \tfrac{1}{24}\, \cancel{g}_{1/2} \\
h_1 - g_1 &= \tfrac{1}{24}\, \cancel{g}_1 \\
h_{3/2} - g_{3/2} &= \tfrac{1}{24}\, \cancel{g}_{3/2} + \tfrac{1}{3456}\, \cancel{g}_{1/2}^3 \\
\ell_2 - g_2 &= \tfrac{1}{24}\, \cancel{g}_2 + \tfrac{5}{9216}\, \cancel{g}_{1/2}^2 \cancel{g}_1 \\
h_{5/2} - g_{5/2} &= \tfrac{1}{24}\, \cancel{g}_{5/2} - \tfrac{1}{9600}\, \cancel{g}_{1/2} \cancel{g}_1^2 - \tfrac{7}{28800}\, \cancel{g}_{1/2}^2 \cancel{g}_{3/2} + \tfrac{1}{1244160}\, \cancel{g}_{1/2}^5 \\
h_3 - g_3 &= \tfrac{1}{24}\, \cancel{g}_3 - \tfrac{1}{17280}\, \cancel{g}_1^3 - \tfrac{1}{1920}\, \cancel{g}_1 \cancel{g}_{3/2} \cancel{g}_{1/2} - \tfrac{1}{2304}\, \cancel{g}_{1/2}^2 \cancel{g}_2 \\
&\quad + \tfrac{1}{497664}\, \cancel{g}_{1/2}^4 \cancel{g}_1 \\
h_{7/2} - g_{7/2} &= \tfrac{1}{24}\, \cancel{g}_{7/2} - \tfrac{53}{201600}\, \cancel{g}_1^2 \cancel{g}_{3/2} - \tfrac{13}{40320}\, \cancel{g}_{1/2} \cancel{g}_{3/2}^2 - \tfrac{11}{14400}\, \cancel{g}_{1/2} \cancel{g}_1 \cancel{g}_2 \\
&\quad - \tfrac{113}{201600}\, \cancel{g}_{1/2}^2 \cancel{g}_{5/2} - \tfrac{11}{21772800}\, \cancel{g}_{1/2}^3 \cancel{g}_1^2 - \tfrac{113}{43545600}\, \cancel{g}_{1/2}^4 \cancel{g}_{3/2} \\
&\quad + \tfrac{1}{836075520}\, \cancel{g}_{1/2}^7 \\
h_4 - g_4 &= \tfrac{1}{24}\, \cancel{g}_4 - \tfrac{1}{1280}\, \cancel{g}_{1/2} \cancel{g}_{3/2} \cancel{g}_2 - \tfrac{7}{23040}\, \cancel{g}_1 \cancel{g}_{3/2}^2 - \tfrac{11}{11520}\, \cancel{g}_{1/2} \cancel{g}_1 \cancel{g}_{5/2} \\
&\quad - \tfrac{1}{2880}\, \cancel{g}_1^2 \cancel{g}_2 - \tfrac{1}{1536}\, \cancel{g}_{1/2}^2 \cancel{g}_3 - \tfrac{229}{159252480}\, \cancel{g}_{1/2}^2 \cancel{g}_1^3 \\
&\quad - \tfrac{229}{39813120}\, \cancel{g}_{1/2}^3 \cancel{g}_1 \cancel{g}_{3/2} - \tfrac{49}{15925248}\, \cancel{g}_{1/2}^4 \cancel{g}_2 + \tfrac{11}{3057647616}\, \cancel{g}_{1/2}^6 \cancel{g}_1 \\
h_{9/2} - g_{9/2} &= \tfrac{1}{24}\, \cancel{g}_{9/2} - \tfrac{43}{60480}\, \cancel{g}_1 \cancel{g}_{3/2} \cancel{g}_2 - \tfrac{67}{60480}\, \cancel{g}_{1/2} \cancel{g}_1 \cancel{g}_3 - \tfrac{17}{40320}\, \cancel{g}_{1/2} \cancel{g}_2^2 \\
&\quad - \tfrac{13}{120960}\, \cancel{g}_{3/2}^3 - \tfrac{29}{40320}\, \cancel{g}_{1/2}^2 \cancel{g}_{7/2} - \tfrac{17}{40320}\, \cancel{g}_1^2 \cancel{g}_{5/2} \\
&\quad - \tfrac{11}{12096}\, \cancel{g}_{1/2} \cancel{g}_{3/2} \cancel{g}_{5/2} + \tfrac{907}{457228800}\, \cancel{g}_{1/2}^2 \cancel{g}_1^2 \cancel{g}_{3/2} \\
&\quad + \tfrac{757}{228614400}\, \cancel{g}_{1/2}^3 \cancel{g}_1 \cancel{g}_2 + \tfrac{79}{914457600}\, \cancel{g}_{1/2} \cancel{g}_1^4 + \tfrac{659}{457228800}\, \cancel{g}_{1/2}^3 \cancel{g}_{3/2}^2 \\
&\quad + \tfrac{1531}{914457600}\, \cancel{g}_{1/2}^4 \cancel{g}_{5/2} - \tfrac{13}{12541132800}\, \cancel{g}_{1/2}^5 \cancel{g}_1^2 \\
&\quad - \tfrac{29}{4180377600}\, \cancel{g}_{1/2}^6 \cancel{g}_{3/2} + \tfrac{1}{902961561600}\, \cancel{g}_{1/2}^9 \\
h_5 - g_5 &= \tfrac{1}{24}\, \cancel{g}_5 - \tfrac{11}{28800}\, \cancel{g}_1 \cancel{g}_2^2 - \tfrac{7}{14400}\, \cancel{g}_1^2 \cancel{g}_3 - \tfrac{47}{57600}\, \cancel{g}_1 \cancel{g}_{3/2} \cancel{g}_{5/2} \\
&\quad - \tfrac{71}{57600}\, \cancel{g}_{1/2} \cancel{g}_1 \cancel{g}_{7/2} - \tfrac{53}{57600}\, \cancel{g}_2 \cancel{g}_{1/2} \cancel{g}_{5/2} - \tfrac{59}{57600}\, \cancel{g}_{1/2} \cancel{g}_{3/2} \cancel{g}_3 \\
&\quad - \tfrac{41}{115200}\, \cancel{g}_{3/2}^2 \cancel{g}_2 - \tfrac{89}{115200}\, \cancel{g}_{1/2}^2 \cancel{g}_4 + \tfrac{1}{21772800}\, \cancel{g}_1^5 \\
&\quad + \tfrac{383}{43545600}\, \cancel{g}_{1/2}^3 \cancel{g}_1 \cancel{g}_{5/2} + \tfrac{797}{116121600}\, \cancel{g}_{1/2}^2 \cancel{g}_1{}^2 \cancel{g}_2 \\
&\quad + \tfrac{361}{174182400}\, \cancel{g}_{1/2} \cancel{g}_1^3 \cancel{g}_{3/2} + \tfrac{41}{6220800}\, \cancel{g}_{1/2}^3 \cancel{g}_{3/2} \cancel{g}_2 \\
&\quad + \tfrac{17}{3225600}\, \cancel{g}_{1/2}^2 \cancel{g}_1 \cancel{g}_{3/2}^2 + \tfrac{19}{4976640}\, \cancel{g}_{1/2}^4 \cancel{g}_3 - \tfrac{1}{188743680}\, \cancel{g}_{1/2}^4 \cancel{g}_1^3 \\
&\quad - \tfrac{47}{2548039680}\, \cancel{g}_{1/2}^5 \cancel{g}_1 \cancel{g}_{3/2} - \tfrac{107}{15288238080}\, \cancel{g}_{1/2}^6 \cancel{g}_2 \\
&\quad + \tfrac{7}{1834588569600}\, \cancel{g}_{1/2}^8 \cancel{g}_1 .
\end{aligned}
$$

10.3.3 Tangency 2, ramification 3

$$
\begin{aligned}
h_{2/3} - g_{2/3} &= \tfrac{1}{24}\, \cancel{g}_{2/3} \\
h_{1} - g_{1} &= \tfrac{1}{24}\, \cancel{g}_{1} \\
h_{4/3} - g_{4/3} &= \tfrac{1}{24}\, \cancel{g}_{4/3} \\
\ell_{5/3} - g_{5/3} &= \tfrac{1}{24}\, \cancel{g}_{5/3} \\
h_{2} - g_{2} &= \tfrac{1}{24}\, \cancel{g}_{2} + \tfrac{1}{5184}\, \cancel{g}_{2/3}^{3} \\
h_{7/3} - g_{7/3} &= \tfrac{1}{24}\, \cancel{g}_{7/3} - \tfrac{1}{40320}\, \cancel{g}_{2/3}^{2} \cancel{g}_{1} \\
h_{8/3} - g_{8/3} &= \tfrac{1}{24}\, \cancel{g}_{8/3} - \tfrac{7}{57600}\, \cancel{g}_{2/3} \cancel{g}_{1}^{2} - \tfrac{1}{5760}\, \cancel{g}_{2/3}^{2} \cancel{g}_{4/3} \\
h_{3} - g_{3} &= \tfrac{1}{24}\, \cancel{g}_{3} - \tfrac{1}{17280}\, \cancel{g}_{1}^{3} - \tfrac{11}{25920}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{4/3} - \tfrac{1}{3456}\, \cancel{g}_{2/3}^{2} \cancel{g}_{5/3} \\
h_{10/3} - g_{10/3} &= \tfrac{1}{24}\, \cancel{g}_{10/3} - \tfrac{53}{201600}\, \cancel{g}_{2/3} \cancel{g}_{4/3}^{2} - \tfrac{47}{201600}\, \cancel{g}_{1}^{2} \cancel{g}_{4/3} \\
&\quad - \tfrac{59}{100800}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{5/3} - \tfrac{11}{28800}\, \cancel{g}_{2/3}^{2} \cancel{g}_{2} + \tfrac{1}{2903040}\, \cancel{g}_{2/3}^{5} \\
h_{11/3} - g_{11/3} &= \tfrac{1}{24}\, \cancel{g}_{11/3} - \tfrac{23}{31680}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{2} - \tfrac{29}{63360}\, \cancel{g}_{2/3}^{2} \cancel{g}_{7/3} - \tfrac{17}{63360}\, \cancel{g}_{1} \cancel{g}_{4/3}^{2} \\
&\quad - \tfrac{37}{126720}\, \cancel{g}_{1}^{2} \cancel{g}_{5/3} - \tfrac{1}{1584}\, \cancel{g}_{2/3} \cancel{g}_{4/3} \cancel{g}_{5/3} + \tfrac{1}{22809600}\, \cancel{g}_{2/3}^{4} \cancel{g}_{1} \\
h_{4} - g_{4} &= \tfrac{1}{24}\, \cancel{g}_{4} - \tfrac{1}{2880}\, \cancel{g}_{1}^{2} \cancel{g}_{2} - \tfrac{1}{1620}\, \cancel{g}_{1} \cancel{g}_{4/3} \cancel{g}_{5/3} - \tfrac{11}{12960}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{7/3} \\
&\quad - \tfrac{1}{2880}\, \cancel{g}_{2/3} \cancel{g}_{5/3}^{2} - \tfrac{1}{10368}\, \cancel{g}_{4/3}^{3} - \tfrac{19}{25920}\, \cancel{g}_{2/3} \cancel{g}_{4/3} \cancel{g}_{2} \\
&\quad - \tfrac{1}{1920}\, \cancel{g}_{2/3}^{2} \cancel{g}_{8/3} - \tfrac{47}{44789760}\, \cancel{g}_{2/3}^{3} \cancel{g}_{1}^{2} - \tfrac{1}{1119744}\, \cancel{g}_{2/3}^{4} \cancel{g}_{4/3} \\
h_{13/3} - g_{13/3} &= \tfrac{1}{24}\, \cancel{g}_{13/3} - \tfrac{131}{187200}\, \cancel{g}_{1} \cancel{g}_{4/3} \cancel{g}_{2} - \tfrac{179}{187200}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{8/3} \\
&\quad - \tfrac{43}{74880}\, \cancel{g}_{2/3}^{2} \cancel{g}_{3} - \tfrac{119}{374400}\, \cancel{g}_{4/3}^{2} \cancel{g}_{5/3} - \tfrac{5}{14976}\, \cancel{g}_{1} \cancel{g}_{5/3}^{2} \\
&\quad - \tfrac{149}{374400}\, \cancel{g}_{1}^{2} \cancel{g}_{7/3} - \tfrac{31}{37440}\, \cancel{g}_{2/3} \cancel{g}_{4/3} \cancel{g}_{7/3} - \tfrac{11}{14400}\, \cancel{g}_{2/3} \cancel{g}_{5/3} \cancel{g}_{2} \\
&\quad - \tfrac{1}{22014720}\, \cancel{g}_{2/3}^{2} \cancel{g}_{1}^{3} + \tfrac{37}{330220800}\, \cancel{g}_{2/3}^{3} \cancel{g}_{1} \cancel{g}_{4/3} \\
&\quad + \tfrac{23}{264176640}\, \cancel{g}_{2/3}^{4} \cancel{g}_{5/3} \\
h_{14/3} - g_{14/3} &= \tfrac{1}{24}\, \cancel{g}_{14/3} - \tfrac{173}{221760}\, \cancel{g}_{1} \cancel{g}_{4/3} \cancel{g}_{7/3} - \tfrac{23}{31680}\, \cancel{g}_{1} \cancel{g}_{5/3} \cancel{g}_{2} \\
&\quad - \tfrac{233}{221760}\, \cancel{g}_{2/3} \cancel{g}_{1} \cancel{g}_{3} - \tfrac{179}{443520}\, \cancel{g}_{2/3} \cancel{g}_{2}^{2} - \tfrac{197}{443520}\, \cancel{g}_{1}^{2} \cancel{g}_{8/3} \\
&\quad - \tfrac{29}{31680}\, \cancel{g}_{2/3} \cancel{g}_{4/3} \cancel{g}_{8/3} - \tfrac{31}{88704}\, \cancel{g}_{4/3}^{2} \cancel{g}_{2} - \tfrac{5}{8064}\, \cancel{g}_{2/3}^{2} \cancel{g}_{10/3} \\
&\quad - \tfrac{149}{443520}\, \cancel{g}_{4/3} \cancel{g}_{5/3}^{2} - \tfrac{37}{44352}\, \cancel{g}_{2/3} \cancel{g}_{5/3} \cancel{g}_{7/3} + \tfrac{1}{9313920}\, \cancel{g}_{2/3} \cancel{g}_{1}^{4} \\
&\quad + \tfrac{1}{776160}\, \cancel{g}_{2/3}^{2} \cancel{g}_{1}^{2} \cancel{g}_{4/3} + \tfrac{1}{1451520}\, \cancel{g}_{2/3}^{3} \cancel{g}_{4/3}^{2} \\
&\quad + \tfrac{17}{10160640}\, \cancel{g}_{2/3}^{3} \cancel{g}_{1} \cancel{g}_{5/3} + \tfrac{23}{31933440}\, \cancel{g}_{2/3}^{4} \cancel{g}_{2} + \tfrac{1}{3065610240}\, \cancel{g}_{2/3}^{7}.
\end{aligned}
$$

10.3.4 Tangency 3, ramification 4

$$
\begin{aligned}
h_{3/4}-g_{3/4} &= \tfrac{1}{24}\, g_{3/4}\\
h_{1}-g_{1} &= \tfrac{1}{24}\, g_{1}\\
h_{5/4}-g_{5/4} &= \tfrac{1}{24}\, g_{5/4}\\
h_{3/2}-g_{3/2} &= \tfrac{1}{24}\, g_{3/2}\\
h_{7/4}-g_{7/4} &= \tfrac{1}{24}\, g_{7/4}\\
h_{2}-g_{2} &= \tfrac{1}{24}\, g_{2}\\
h_{9/4}-g_{9/4} &= \tfrac{1}{24}\, g_{9/4} + \tfrac{1}{86400}\, g_{3/4}^{3}\\
h_{5/2}-g_{5/2} &= \tfrac{1}{24}\, g_{5/2} - \tfrac{1}{14400}\, g_{3/4}^{2} g_{1}\\
h_{11/4}-g_{11/4} &= \tfrac{1}{24}\, g_{11/4} - \tfrac{59}{443520}\, g_{3/4} g_{1}^{2} - \tfrac{71}{443520}\, g_{3/4}^{2} g_{5/4}\\
h_{3}-g_{3} &= \tfrac{1}{24}\, g_{3} - \tfrac{1}{17280}\, g_{1}^{3} - \tfrac{1}{2560}\, g_{3/4} g_{1} g_{5/4} - \tfrac{11}{46080}\, g_{3/4}^{2} g_{3/2}\\
h_{13/4}-g_{13/4} &= \tfrac{1}{24}\, g_{13/4} - \tfrac{49}{224640}\, g_{1}^{2} g_{5/4} - \tfrac{19}{37440}\, g_{3/4} g_{1} g_{3/2}\\
&\quad - \tfrac{53}{224640}\, g_{3/4} g_{5/4}^{2} - \tfrac{23}{74880}\, g_{3/4}^{2} g_{7/4}\\
h_{7/2}-g_{7/2} &= \tfrac{1}{24}\, g_{7/2} - \tfrac{53}{201600}\, g_{1}^{2} g_{3/2} - \tfrac{31}{50400}\, g_{3/4} g_{1} g_{7/4}\\
&\quad - \tfrac{1}{1800}\, g_{3/4} g_{5/4} g_{3/2} - \tfrac{37}{100800}\, g_{3/4}^{2} g_{2} - \tfrac{1}{4032}\, g_{1} g_{5/4}^{2}\\
h_{15/4}-g_{15/4} &= \tfrac{1}{24}\, g_{15/4} - \tfrac{113}{158400}\, g_{3/4} g_{1} g_{2} - \tfrac{89}{158400}\, g_{3/2} g_{1} g_{5/4}\\
&\quad - \tfrac{17}{190080}\, g_{5/4}^{3} - \tfrac{97}{316800}\, g_{3/4} g_{3/2}^{2} - \tfrac{97}{316800}\, g_{1}^{2} g_{7/4}\\
&\quad - \tfrac{101}{158400}\, g_{3/4} g_{5/4} g_{7/4} - \tfrac{133}{316800}\, g_{3/4}^{2} g_{9/4}\\
&\quad + \tfrac{1}{53222400}\, g_{3/4}^{5}\\
h_{4}-g_{4} &= \tfrac{1}{24}\, g_{4} - \tfrac{1}{2880}\, g_{1}^{2} g_{2} - \tfrac{7}{23040}\, g_{1} g_{3/2}^{2} - \tfrac{37}{46080}\, g_{3/4} g_{1} g_{9/4}\\
&\quad - \tfrac{29}{46080}\, g_{1} g_{5/4} g_{7/4} - \tfrac{43}{92160}\, g_{3/4}^{2} g_{5/2} - \tfrac{11}{15360}\, g_{3/4} g_{5/4} g_{2}\\
&\quad - \tfrac{3}{10240}\, g_{5/4}^{2} g_{3/2} - \tfrac{31}{46080}\, g_{3/4} g_{3/2} g_{7/4} - \tfrac{29}{70778880}\, g_{3/4}^{4} g_{1}\\
h_{17/4}-g_{17/4} &= \tfrac{1}{24}\, g_{17/4} - \tfrac{563}{636480}\, g_{3/4} g_{1} g_{5/2} - \tfrac{419}{636480}\, g_{3/2} g_{1} g_{7/4}\\
&\quad - \tfrac{443}{636480}\, g_{1} g_{5/4} g_{2} - \tfrac{7}{19584}\, g_{3/4} g_{7/4}^{2} - \tfrac{407}{1272960}\, g_{5/4}^{2} g_{7/4}\\
&\quad - \tfrac{467}{636480}\, g_{3/4} g_{3/2} g_{2} - \tfrac{79}{254592}\, g_{5/4} g_{3/2}^{2} - \tfrac{647}{1272960}\, g_{3/4}^{2} g_{11/4}\\
&\quad - \tfrac{491}{1272960}\, g_{1}^{2} g_{9/4} - \tfrac{503}{636480}\, g_{3/4} g_{5/4} g_{9/4} - \tfrac{1}{7128576}\, g_{3/4}^{3} g_{1}^{2}\\
&\quad - \tfrac{43}{641571840}\, g_{3/4}^{4} g_{5/4}.
\end{aligned}
$$

10.4 The *nir* transform: from f to h

In all the tables that follow, the vertical bars |||||| separate clusters of terms with different homogeneous degree in f.

10.4.1 Tangency 0, ramification 1

$$h_0 = f_0^{-1}$$

$$h_1 = -f_0^{-3} f_1 \ \ |||||| + \tfrac{1}{24} f_0^{-1} f_1$$

$$h_2 = +\tfrac{3}{2} f_0^{-5} f_1^2 - f_0^{-4} f_2 \ |||||| - \tfrac{1}{48} f_0^{-3} f_1^2 + \tfrac{1}{24} f_0^{-2} f_2 \ |||||| + \tfrac{1}{2304} f_0^{-1} f_1^2$$

$$\begin{aligned}h_3 = & -\tfrac{5}{2} f_0^{-7} f_1^3 + \tfrac{10}{3} f_0^{-6} f_1 f_2 - f_0^{-5} f_3 \ \ |||||| + \tfrac{1}{48} f_0^{-5} f_1^3 - \tfrac{1}{18} f_0^{-4} f_1 f_2 \\ & + \tfrac{1}{24} f_0^{-3} f_3 \ \ |||||| - \tfrac{1}{6912} f_0^{-3} f_1^3 + \tfrac{1}{1728} f_0^{-2} f_1 f_2 - \tfrac{7}{5760} f_0^{-1} f_3 \ \ |||||| \\ & + \tfrac{1}{497664} f_0^{-1} f_1^3\end{aligned}$$

$$\begin{aligned}h_4 = & \tfrac{35}{8} f_0^{-9} f_1^4 - \tfrac{35}{4} f_0^{-8} f_1^2 f_2 + \tfrac{5}{3} f_0^{-7} f_2^2 + \tfrac{15}{4} f_0^{-7} f_1 f_3 - f_0^{-6} f_4 \ \ |||||| \\ & - \tfrac{5}{192} f_0^{-7} f_1^4 + \tfrac{25}{288} f_0^{-6} f_1^2 f_2 - \tfrac{7}{96} f_0^{-5} f_1 f_3 - \tfrac{1}{36} f_0^{-5} f_2^2 \\ & + \tfrac{1}{24} f_0^{-4} f_4 \ \ |||||| + \tfrac{1}{9216} f_0^{-5} f_1^4 - \tfrac{7}{13824} f_0^{-4} f_1^2 f_2 + \tfrac{17}{23040} f_0^{-3} f_1 f_3 \\ & + \tfrac{1}{3456} f_0^{-3} f_2^2 - \tfrac{7}{5760} f_0^{-2} f_4 |||||| - \tfrac{1}{1990656} f_0^{-3} f_1^4 + \tfrac{1}{331776} f_0^{-2} f_1^2 f_2 \\ & - \tfrac{7}{552960} f_0^{-1} f_1 f_3 |||||| + \tfrac{1}{191102976} f_0^{-1} f_1^4\end{aligned}$$

$$\begin{aligned}h_5 = & -\tfrac{63}{8} f_0^{-11} f_1^5 + 21 f_0^{-10} f_1^3 f_2 - \tfrac{28}{3} f_0^{-9} f_1 f_2^2 - \tfrac{21}{2} f_0^{-9} f_1^2 f_3 \\ & + \tfrac{7}{2} f_0^{-8} f_2 f_3 + \tfrac{21}{5} f_0^{-8} f_1 f_4 - f_0^{-7} f_5 \ \ |||||| + \tfrac{7}{192} f_0^{-9} f_1^5 \\ & - \tfrac{7}{48} f_0^{-8} f_1^3 f_2 + \tfrac{1}{8} f_0^{-7} f_1^2 f_3 + \tfrac{7}{72} f_0^{-7} f_1 f_2^2 - \tfrac{1}{16} f_0^{-6} f_2 f_3 \\ & - \tfrac{11}{120} f_0^{-6} f_1 f_4 + \tfrac{1}{24} f_0^{-5} f_5 \ \ |||||| - \tfrac{1}{9216} f_0^{-7} f_1^5 + \tfrac{1}{1728} f_0^{-6} f_1^3 f_2 \\ & - \tfrac{43}{57600} f_0^{-5} f_1^2 f_3 - \tfrac{1}{1728} f_0^{-5} f_1 f_2^2 + \tfrac{37}{57600} f_0^{-4} f_2 f_3 \\ & + \tfrac{31}{28800} f_0^{-4} f_1 f_4 - \tfrac{7}{5760} f_0^{-3} f_5 \ \ |||||| + \tfrac{1}{3317760} f_0^{-5} f_1^5 \\ & - \tfrac{1}{497664} f_0^{-4} f_1^3 f_2 + \tfrac{1}{230400} f_0^{-3} f_1^2 f_3 + \tfrac{1}{414720} f_0^{-3} f_1 f_2^2 \\ & - \tfrac{7}{1382400} f_0^{-2} f_2 f_3 - \tfrac{7}{691200} f_0^{-2} f_1 f_4 \\ & + \tfrac{31}{967680} f_0^{-1} f_5 \ \ |||||| - \tfrac{1}{955514880} f_0^{-3} f_1^5 + \tfrac{1}{119439360} f_0^{-2} f_1^3 f_2 \\ & - \tfrac{7}{132710400} f_0^{-1} f_1^2 f_3 \ \ |||||| + \tfrac{1}{114661785600} f_0^{-1} f_1^5.\end{aligned}$$

10.4.2 Tangency 1, ramification 2

$$h_{-1/2}=2^{-1/2}\Big\{ f_1^{-1/2} \Big\}$$

$$h_0 = \Big\{ -\tfrac{2}{3} f_1^{-2} f_2 \Big\}$$

$$h_{1/2} = 2^{1/2}\Big\{ \tfrac{5}{6} f_1^{-7/2} f_2^2 - \tfrac{3}{4} f_1^{-5/2} f_3 \;\Vert\Vert\Vert\; + \tfrac{1}{24} f_1^{1/2} \Big\}$$

$$h_1 = 2 \Big\{ -\tfrac{4}{5} f_1^{-3} f_4 - \tfrac{32}{27} f_1^{-5} f_2^3 + 2 f_1^{-4} f_2 f_3 \;\Vert\Vert\Vert\; + \tfrac{1}{36} f_1^{-1} f_2 \Big\}$$

$$\begin{aligned}
h_{3/2} = 2^{3/2}\Big\{ & -\tfrac{5}{6} f_1^{-7/2} f_5 + \tfrac{7}{3} f_1^{-9/2} f_2 f_4 - \tfrac{35}{8} f_1^{-11/2} f_2^2 f_3 \\
& + \tfrac{385}{216} f_1^{-13/2} f_2^4 + \tfrac{35}{32} f_1^{-9/2} f_3^2 \;\Vert\Vert\Vert\; - \tfrac{7}{432} f_1^{-5/2} f_2^2 \\
& + \tfrac{1}{32} f_1^{-3/2} f_3 \;\Vert\Vert\Vert\; + \tfrac{1}{3456} f_1^{3/2} \Big\}
\end{aligned}$$

$$\begin{aligned}
h_2 = 2^2 \Big\{ & -\tfrac{16}{3} f_1^{-6} f_2^2 f_4 - 5 f_1^{-6} f_2 f_3^2 + \tfrac{80}{9} f_1^{-7} f_2^3 f_3 + \tfrac{8}{3} f_1^{-5} f_2 f_5 \\
& + \tfrac{12}{5} f_1^{-5} f_3 f_4 - \tfrac{6}{7} f_1^{-4} f_6 - \tfrac{224}{81} f_1^{-8} f_2^5 \;\Vert\Vert\Vert\; + \tfrac{1}{30} f_1^{-2} f_4 \\
& + \tfrac{5}{324} f_1^{-4} f_2^3 - \tfrac{1}{24} f_1^{-3} f_2 f_3 \;\Vert\Vert\Vert\; + \tfrac{5}{13824} f_2 \Big\}
\end{aligned}$$

$$\begin{aligned}
h_{5/2} = 2^{5/2}\Big\{ & -\tfrac{231}{20} f_1^{-13/2} f_2 f_3 f_4 - \tfrac{7}{8} f_1^{-9/2} f_7 - \tfrac{5005}{288} f_1^{-17/2} f_2^4 f_3 \\
& + \tfrac{1001}{90} f_1^{-15/2} f_2^3 f_4 - \tfrac{231}{128} f_1^{-13/2} f_3^3 + \tfrac{17017}{3888} f_1^{-19/2} f_2^6 \\
& - \tfrac{77}{12} f_1^{-13/2} f_5 f_2^2 + \tfrac{21}{8} f_1^{-11/2} f_3 f_5 + \tfrac{63}{50} f_1^{-11/2} f_4^2 \\
& + \tfrac{1001}{64} f_1^{-15/2} f_2^2 f_3^2 + 3 f_1^{-11/2} f_2 f_6 \;\Vert\Vert\Vert\; - \tfrac{91}{5184} f_1^{-11/2} f_2^4 \\
& + \tfrac{5}{144} f_1^{-5/2} f_5 - \tfrac{17}{768} f_1^{-7/2} f_3^2 + \tfrac{35}{576} f_1^{-9/2} f_2^2 f_3 \\
& - \tfrac{19}{360} f_1^{-7/2} f_2 f_4 \;\Vert\Vert\Vert\; - \tfrac{7}{38400} f_1^{-1/2} f_3 + \tfrac{1}{20736} f_1^{-3/2} f_2^2 \\
& \;\Vert\Vert\Vert\; \tfrac{1}{1244160} f_1^{5/2} \Big\}
\end{aligned}$$

$$\begin{aligned}
h_3 = 2^3 \Big\{ & \tfrac{112}{3} f_1^{-8} f_2^2 f_3 f_4 - \tfrac{40}{3} f_1^{-7} f_2 f_3 f_5 + \tfrac{10}{3} f_1^{-6} f_2 f_7 + \tfrac{20}{7} f_1^{-6} f_3 f_6 \\
& - \tfrac{160}{21} f_1^{-7} f_2^2 f_6 + \tfrac{8}{3} f_1^{-6} f_4 f_5 + \tfrac{1120}{81} f_1^{-8} f_2^3 f_5 - \tfrac{32}{5} f_1^{-7} f_2 f_4^2 \\
& - \tfrac{8}{9} f_1^{-5} f_8 - 6 f_1^{-7} f_3^2 f_4 - \tfrac{1792}{81} f_1^{-9} f_2^4 f_4 - \tfrac{1120}{27} f_1^{-9} f_2^3 f_3^2 \\
& + \tfrac{35}{3} f_1^{-8} f_2 f_3^3 - \tfrac{5120}{729} f_1^{-11} f_2^7 + \tfrac{896}{27} f_1^{-10} f_2^5 f_3 \;\Vert\Vert\Vert \\
& + \tfrac{1}{28} f_1^{-3} f_6 - \tfrac{7}{108} f_1^{-4} f_2 f_5 - \tfrac{5}{54} f_1^{-6} f_2^3 f_3 + \tfrac{16}{729} f_1^{-7} f_2^5 \\
& + \tfrac{11}{135} f_1^{-5} f_2^2 f_4 + \tfrac{5}{72} f_1^{-5} f_2 f_3^2 - \tfrac{1}{20} f_1^{-4} f_3 f_4 \;\Vert\Vert\Vert \\
& - \tfrac{1}{23328} f_1^{-3} f_2^3 - \tfrac{1}{2880} f_1^{-1} f_4 + \tfrac{1}{5760} f_1^{-2} f_2 f_3 \\
& \;\Vert\Vert\Vert\; + \tfrac{1}{746496} f_1 f_2 \Big\}.
\end{aligned}$$

10.4.3 Tangency 2, ramification 3

$$h_{-2/3}=3^{-2/3}\left\{f_2^{-1/3}\right\}$$

$$h_{-1/3}=3^{-1/3}\left\{-\tfrac{1}{2}\,f_2^{-5/3}f_3\right\}$$

$$h_0\quad=\quad\left\{\tfrac{9}{16}f_2^{-3}f_3^2-\tfrac{3}{5}f_2^{-2}f_4\right\}$$

$$h_{1/3}\;=3^{1/3}\left\{-\tfrac{2}{3}\,f_2^{-7/3}f_5-\tfrac{35}{48}\,f_2^{-13/3}f_3^3+\tfrac{7}{5}\,f_2^{-10/3}f_3f_4\right\}$$

$$\begin{aligned}h_{2/3}\;=3^{2/3}\Big\{&\tfrac{385}{384}\,f_2^{-17/3}f_3^4+\tfrac{5}{3}\,f_2^{-11/3}f_3f_5+\tfrac{4}{5}\,f_2^{-11/3}f_4^2-\tfrac{5}{7}\,f_2^{-8/3}f_6\\&-\tfrac{11}{4}\,f_2^{-14/3}f_3^2f_4\quad||||||+\tfrac{1}{24}\,f_2^{1/3}\Big\}\end{aligned}$$

$$\begin{aligned}h_1\quad=3\Big\{&\tfrac{9}{5}\,f_2^{-4}f_4f_5+\tfrac{81}{16}\,f_2^{-6}f_4f_3^3-\tfrac{3}{4}\,f_2^{-3}f_7+\tfrac{27}{14}\,f_2^{-4}f_3f_6\\&-\tfrac{27}{8}\,f_2^{-5}f_3^2f_5-\tfrac{81}{25}\,f_2^{-5}f_3f_4^2-\tfrac{729}{512}\,f_2^{-7}f_3^5||||||+\tfrac{1}{48}\,f_3f_2^{-1}\Big\}\end{aligned}$$

$$\begin{aligned}h_{4/3}\;=3^{4/3}\Big\{&-\tfrac{91}{75}\,f_2^{-16/3}f_4^3-\tfrac{91}{12}\,f_2^{-16/3}f_3f_4f_5-\tfrac{7}{9}\,f_2^{-10/3}f_8\\&-\tfrac{65}{16}\,f_2^{-16/3}f_3^2f_6+\tfrac{35}{16}\,f_2^{-13/3}f_3f_7+2\,f_2^{-13/3}f_4f_6\\&-\tfrac{1729}{192}\,f_2^{-22/3}f_3^4f_4+\tfrac{19019}{9216}\,f_2^{-25/3}f_3^6\\&+\tfrac{455}{72}\,f_2^{-19/3}f_3^3f_5+\tfrac{35}{36}\,f_2^{-13/3}{f_5}^2+\tfrac{91}{10}\,f_2^{-19/3}f_3^2f_4^2\\&\;||||||+\tfrac{1}{40}\,f_2^{-4/3}f_4-\tfrac{5}{384}\,f_2^{-7/3}f_3^2\Big\}\end{aligned}$$

$$\begin{aligned}h_{5/3}\;=3^{5/3}\Big\{&-\tfrac{4}{5}\,f_2^{-11/3}f_9-\tfrac{44}{5}\,f_2^{-17/3}f_4f_6f_3-\tfrac{77}{16}\,f_2^{-17/3}f_3^2f_7\\&-\tfrac{1309}{60}\,f_2^{-23/3}f_3^3f_4^2-\tfrac{6545}{576}\,f_2^{-23/3}f_3^4f_5+\tfrac{187}{24}\,f_2^{-20/3}f_3^3f_6\\&+\tfrac{30107}{1920}\,f_2^{-26/3}f_3^5f_4-\tfrac{55913}{18432}\,f_2^{-29/3}f_3^7+\tfrac{1309}{60}\,f_2^{-20/3}f_3^2f_4f_5\\&+\tfrac{44}{21}\,f_2^{-14/3}f_6f_5+\tfrac{2618}{375}\,f_2^{-20/3}f_3f_4^3+\tfrac{22}{9}\,f_2^{-14/3}f_3f_8\\&+\tfrac{11}{5}\,f_2^{-14/3}f_7f_4-\tfrac{308}{75}\,f_2^{-17/3}f_4^2f_5-\tfrac{77}{18}\,f_2^{-17/3}{f_5}^2f_3||||||\\&-\tfrac{1}{30}\,f_2^{-8/3}f_3f_4+\tfrac{7}{576}\,f_2^{-11/3}f_3^3+\;1/36\,f_2^{-5/3}f_5\Big\}\end{aligned}$$

$$\begin{aligned}h_2\quad=3^2\Big\{&\tfrac{729}{28}\,f_2^{-7}f_3^2f_4f_6-\tfrac{135}{14}\,f_2^{-6}f_3f_5f_6-\tfrac{1701}{32}\,f_2^{-8}f_3^3f_4f_5\\&+\tfrac{243}{10}\,f_2^{-7}f_3f_4^2f_5-\tfrac{9}{2}\,f_2^{-6}{f_5}^2f_4-\tfrac{81}{8}\,f_2^{-6}f_3f_4f_7\\&+\tfrac{295245}{65536}f_2^{-11}f_3^8+\tfrac{27}{10}f_2^{-5}f_3f_9-\tfrac{3645}{256}f_2^{-8}f_3^4f_6+\tfrac{243}{125}\,f_2^{-7}f_4^4\\&-\tfrac{9}{11}\,f_2^{-4}f_{10}-\tfrac{137781}{5120}\,f_2^{-10}f_3^6f_4+\tfrac{54}{49}\,f_2^{-5}{f_6}^2\\&-\tfrac{45}{8}\,f_2^{-6}f_3^2f_8-\tfrac{5103}{200}\,f_2^{-8}f_3^2f_4^3+\tfrac{5103}{256}\,f_2^{-9}f_3^5f_5\\&+\tfrac{15309}{320}f_2^{-9}f_3^4f_4^2+\tfrac{12}{5}f_2^{-5}f_8f_4+\tfrac{405}{32}f_2^{-7}f_3^2{f_5}^2-\tfrac{162}{35}f_2^{-6}f_4^2f_6\\&+\tfrac{1215}{128}f_2^{-7}f_3^3f_7+\tfrac{9}{4}f_2^{-5}f_5f_7\;||||||-\tfrac{27}{2048}f_2^{-5}f_3^4-\tfrac{1}{24}f_2^{-3}f_3f_5\\&+\tfrac{5}{168}\,f_2^{-2}f_6-\tfrac{11}{600}\,f_2^{-3}f_4^2+\tfrac{3}{64}\,f_2^{-4}f_3^2f_4\quad||||||+\tfrac{1}{5184}\,f_2^{-1}\Big\}.\end{aligned}$$

10.4.4 Tangency 3, ramification 4

$$
\begin{aligned}
h_{-3/4} &= 4^{-3/4}\Big\{ f_3^{-3/4} \Big\}\\
h_{-1/2} &= 4^{-1/2}\Big\{ -\tfrac{2}{5}\, f_3^{-3/2} f_4 \Big\}\\
h_{-1/4} &= 4^{-1/4}\Big\{ \tfrac{21}{50}\, f_3^{-11/4} f_4^2 - \tfrac{1}{2}\, f_3^{-7/4} f_5 \Big\}\\
h_0 &= \Big\{ -\tfrac{64}{125} f_3^{-4} f_4^3 + \tfrac{16}{15} f_3^{-3} f_4 f_5 - \tfrac{4}{7} f_3^{-2} f_6 \Big\}\\
h_{1/4} &= 4^{1/4}\Big\{ -\tfrac{39}{20}\, f_3^{-17/4} f_4^2 f_5 + \tfrac{9}{7}\, f_3^{-13/4} f_4 f_6 + \tfrac{5}{8}\, f_3^{-13/4} f_5{}^2\\
&\quad +\tfrac{663}{1000}\, f_3^{-21/4} f_4^4 - \tfrac{5}{8}\, f_3^{-9/4} f_7 \Big\}\\
h_{1/2} &= 4^{1/2}\Big\{ \tfrac{84}{25}\, f_3^{-11/2} f_4^3 f_5 - \tfrac{2772}{3125}\, f_3^{-13/2} f_4^5 - \tfrac{12}{5}\, f_3^{-9/2} f_4^2 f_6\\
&\quad -\tfrac{7}{3} f_3^{-9/2} f_4 f_5{}^2 + \tfrac{3}{2} f_3^{-7/2} f_4 f_7 + \tfrac{10}{7} f_3^{-7/2} f_5 f_6 - \tfrac{2}{3} f_3^{-5/2} f_8 \Big\}\\
h_{3/4} &= 4^{3/4}\Big\{ -\tfrac{231}{80}\, f_3^{-19/4} f_4^2 f_7 - \tfrac{11}{2}\, f_3^{-19/4} f_4 f_5 f_6 - \tfrac{385}{432}\, f_3^{-19/4} f_5{}^3\\
&\quad +\tfrac{209}{50}\, f_3^{-23/4} f_4^3 f_6 + \tfrac{1463}{240}\, f_3^{-23/4} f_4^2 f_5{}^2 - \tfrac{7}{10}\, f_3^{-11/4} f_9\\
&\quad +\tfrac{11}{14}\, f_3^{-15/4} f_6{}^2 + \tfrac{302841}{250000}\, f_3^{-31/4} f_4^6 + \tfrac{77}{45}\, f_3^{-15/4} f_4 f_8\\
&\quad +\tfrac{77}{48}\, f_3^{-15/4} f_7 f_5 - \tfrac{33649}{6000}\, f_3^{-27/4} f_5 f_4^4 \;\;\|\|\|\;\; + \tfrac{1}{24}\, f_3^{1/4} \Big\}\\
h_1 &= 4\;\;\Big\{ +\tfrac{28672}{3125} f_3^{-8} f_4^5 f_5 - \tfrac{131072}{78125} f_3^{-9} f_4^7 - \tfrac{8}{11} f_3^{-3} f_{10} - \tfrac{32}{5} f_3^{-5} f_4 f_5 f_7\\
&\quad -\tfrac{256}{75} f_3^{-5} f_4^2 f_8 - \tfrac{768}{245} f_3^{-5} f_4 f_6{}^2 - \tfrac{64}{21} f_3^{-5} f_5{}^2 f_6 + \tfrac{48}{25} f_3^{-4} f_4 f_9\\
&\quad +\tfrac{16}{9} f_3^{-4} f_5 f_8 + \tfrac{12}{7} f_3^{-4} f_6 f_7 - \tfrac{6144}{875} f_3^{-7} f_4^4 f_6 - \tfrac{1024}{75} f_3^{-7} f_4^3 f_5{}^2\\
&\quad +\tfrac{512}{35}\, f_3^{-6} f_4^2 f_5 f_6 + \tfrac{128}{25}\, f_3^{-6} f_4^3 f_7 + \tfrac{128}{27}\, f_3^{-6} f_4 f_5{}^3 \;\;\|\|\|\\
&\quad +\tfrac{1}{60} f_3^{-1} f_4 \Big\}\\
h_{5/4} &= 4^{5/4}\Big\{ \tfrac{23476167}{10000000}\, f_3^{-41/4} f_4^8 + \tfrac{13}{7}\, f_3^{-17/4} f_6 f_8 + \tfrac{57681}{5000}\, f_3^{-33/4} f_4^5 f_6\\
&\quad +\tfrac{1547}{1152}\, f_3^{-25/4} f_5{}^4 - \tfrac{1989}{500}\, f_3^{-21/4} f_4^2 f_9 - \tfrac{221}{64}\, f_3^{-21/4} f_5{}^2 f_7\\
&\quad -\tfrac{663}{196}\, f_3^{-21/4} f_5 f_6{}^2 - \tfrac{1480479}{100000} f_3^{-37/4} f_4^6 f_5 + \tfrac{5967}{700} f_3^{-25/4} f_4^2 f_6{}^2\\
&\quad -\tfrac{13923}{1600}\, f_3^{-29/4} f_4^4 f_7 - \tfrac{1547}{96}\, f_3^{-29/4} f_4^2 f_5{}^3 + \tfrac{117}{55}\, f_3^{-17/4} f_4 f_{10}\\
&\quad +\tfrac{39}{20}\, f_3^{-17/4} f_9 f_5 + \tfrac{1547}{250}\, f_3^{-25/4} f_4^3 f_8 + \tfrac{117}{128}\, f_3^{-17/4} f_7{}^2\\
&\quad -\tfrac{3}{4}\, f_3^{-13/4} f_{11} - \tfrac{663}{20}\, f_3^{-29/4} f_4^3 f_5 f_6 + \tfrac{663}{40}\, f_3^{-25/4} f_5{}^2 f_6 f_4\\
&\quad +\tfrac{44863}{1600}\, f_3^{-33/4} f_4^4 f_5{}^2 - \tfrac{1989}{280}\, f_3^{-21/4} f_4 f_6 f_7\\
&\quad +\tfrac{13923}{800}\, f_3^{-25/4} f_4^2 f_5 f_7 - \tfrac{221}{30}\, f_3^{-21/4} f_4 f_5 f_8 \;\;\|\|\|\\
&\quad +\tfrac{1}{48}\, f_3^{-5/4} f_5 - \tfrac{13}{1200}\, f_3^{-9/4} f_4^2 \Big\}.
\end{aligned}
$$

10.5 The *nur* transform: from f to h

10.5.1 Tangency 0

$$h_0 = f_0{}^{<-1>}$$

$$h_1 = -f_0{}^{<-3>} f_1 \;\; \|\|\|\| + \tfrac{1}{24} f_0{}^{<-1>} f_1$$

$$h_2 = +\tfrac{3}{2} f_0{}^{<-5>} f_1^2 - f_0{}^{<-4>} f_2 \;\; \|\|\|\| - \tfrac{1}{48} f_0{}^{<-3>} f_1^2$$

$$+\tfrac{1}{24} f_0{}^{<-2>} f_2 \;\; \|\|\|\| + \tfrac{1}{2304} f_0{}^{<-1>} f_1^2$$

....

$$\alpha^{<-k>} := \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(\alpha+2\pi\, i\, n)^k} = \frac{1}{(k-1)!}\left[\partial_\sigma^{k-1} \frac{2}{\sinh\left(\dfrac{\alpha-\sigma}{2}\right)}\right]_{\sigma=0}$$

$$\equiv (-1)^k\,(-\alpha)^{<-k>}$$

$$\alpha^{<-1>} = \frac{\sqrt{a}}{a-1} \qquad \text{with} \quad a := e^\alpha$$

$$\alpha^{<-2>} = \frac{\sqrt{a}(a+1)}{2\,(a-1)^2}$$

$$\alpha^{<-3>} = \frac{\sqrt{a}\left(a^2+6\,a+1\right)}{8\,(a-1)^3}$$

$$\alpha^{<-4>} = \frac{\sqrt{a}\left(a^3+23\,a^2+23\,a+1\right)}{48\,(a-1)^4}$$

$$\alpha^{<-5>} = \frac{\sqrt{a}\left(a^4+76\,a^3+230\,a^2+76\,a+1\right)}{384\,(a-1)^5}$$

$$\alpha^{<-6>} = \frac{\sqrt{a}\left(a^5+237\,a^4+1682\,a^3+1682\,a^2+237\,a+1\right)}{3840\,(a-1)^6}$$

$$\alpha^{<-7>} = \frac{\sqrt{a}\left(a^6+722\,a^5+10543\,a^4+23548\,a^3+10543\,a^2+722\,a+1\right)}{46080\,(a-1)^7}$$

$$\alpha^{<-8>} = \frac{\sqrt{a}\left(a^7+2179\,a^6+60657\,a^5+259723\,a^4+259723\,a^3+60657\,a^2+2179\,a+1\right)}{645120\,(a-1)^8}$$

$$\alpha^{<-9>} = \frac{\sqrt{a}(a^8+6552a^7+331612a^6+2485288a^5+4675014a^4+2485288a^3+331612a^2+6552a+1)}{10321920(a-1)^9}.$$

....

10.5.2 Tangency > 0 We set $Dh := nur(f) - nir(f)$ and to calculate the general Dh_n, we take h_n in the *nir*-table Section 10.4 and perform the substitution:

$$f_0^{-k} \longrightarrow \sum_{n\in\mathbb{Z}^*}\frac{1}{(2\pi\, i)^k} = -\beta_{k-1} = \frac{1}{(k-1)!}\left[\partial_\sigma^{k-1}\frac{2}{\sinh\left(-\frac{\sigma}{2}\right)}\right]_{\sigma=0}$$

$$Dh_0 = Dh_1 = 0$$

$$Dh_2 = +\tfrac{17}{5760}\, f_2 \qquad \equiv \left(\beta_3 - \tfrac{1}{24}\,\beta_1\right)\, f_2$$

$$Dh_3 = +\tfrac{1}{5040}\, f_2 f_1 \qquad \equiv \left(-\tfrac{10}{3}\,\beta_5 + \tfrac{1}{18}\,\beta_3 - \tfrac{1}{1728}\,\beta_1\right) f_1 f_2$$

$$Dh_4 = -\tfrac{43}{322560}\, f_4 + \tfrac{11927}{1114767360}\, f_2 f_1^2$$

$$Dh_5 = -\tfrac{18839}{2322432000}\, f_4 f_1 - \tfrac{27241}{4644864000}\, f_3 f_2 + \tfrac{28283}{200658124800}\, f_2 f_1^3$$

$$\begin{aligned} Dh_6 = {} & +\tfrac{769}{154828800}\, f_6 - \tfrac{143}{746496000}\, f_4 f_1^2 - \tfrac{28709}{111476736000}\, f_3 f_2 f_1 \\ & -\tfrac{677}{20065812480}\, f_2^3 + \tfrac{1657}{2407897497600}\, f_2 f_1^4 \end{aligned}$$

$$\begin{aligned} Dh_7 = {} & +\tfrac{319}{1560674304}\, f_6 f_1 + \tfrac{13277}{117050572800}\, f_5 f_2 + \tfrac{373}{4335206400}\, f_4 f_3 \\ & -\tfrac{41539}{28092137472000}\, f_4 f_1^3 - \tfrac{156581}{56184274944000}\, f_3 f_2 f_1^2 - \tfrac{667}{936404582400}\, f_2^3 f_1 \\ & +\tfrac{6133}{3467372396544000}\, f_2 f_1^5 \end{aligned}$$

$$\begin{aligned} Dh_8 = {} & -\tfrac{163}{1114767360}\, f_8 + \tfrac{32843}{10487731322880}\, f_6 f_1^2 + \tfrac{27151}{8739776102400}\, f_5 f_2 f_1 \\ & +\tfrac{207481}{93640458240000}\, f_4 f_3 f_1 + \tfrac{2693}{3121348608000}\, f_4 f_2^2 + \tfrac{269489}{374561832960000}\, f_3^2 f_2 \\ & -\tfrac{32447}{6472428473548800}\, f_4 f_1^4 - \tfrac{6953}{577895399424000}\, f_3 f_2 f_1^3 \\ & -\tfrac{7157}{1618107118387200}\, f_2^3 f_1^2 + \tfrac{633697}{2236871280458465280000}\, f_2 f_1^6. \end{aligned}$$

....

10.6 Translocation of *nir*

10.6.1 Standard case

$$(\delta_1\, h)_0 = +\tfrac{1}{24}\, f_1$$

$$(\delta_1\, h)_1 = +\tfrac{1}{1152}\, f_1^2$$

$$(\delta_1\, h)_2 = -\tfrac{7}{1920}\, f_3 + \tfrac{1}{165888}\, f_1^3$$

$$(\delta_1\, h)_3 = -\tfrac{7}{138240}\, f_1 f_3 + \tfrac{1}{47775744}\, f_1^4$$

$$(\delta_1\, h)_4 = +\tfrac{31}{193536}\, f_5 - \tfrac{7}{26542080}\, f_3 f_1^2 + \tfrac{1}{22932357120}\, f_1^5$$

$$(\delta_1\, h)_5 = +\tfrac{49}{221184000}\, f_3^2 + \tfrac{31}{23224320}\, f_1 f_5 - \tfrac{7}{9555148800}\, f_1^3 f_3 + \tfrac{1}{16511297126400}\, f_1^6$$

$$\begin{aligned} (\delta_1\, h)_6 = {} & -\tfrac{127}{22118400}\, f_7 + \tfrac{31}{6688604160}\, f_1^2 f_5 + \tfrac{49}{31850496000}\, f_1 f_3^2 \\ & -\tfrac{7}{5503765708800}\, f_1^4 f_3 + \tfrac{1}{16643387503411200}\, f_1^7 \end{aligned}$$

$$
\begin{aligned}
(\delta_2\, h)_0 =& +\tfrac{1}{24}\, f_2 - \tfrac{1}{2304}\, f_0\, f_1^2\\
(\delta_2\, h)_1 =& +\tfrac{1}{576}\, f_1\, f_2 + \tfrac{7}{1920}\, f_0\, f_3 - \tfrac{1}{165888}\, f_0\, f_1^3\\
(\delta_2\, h)_2 =& -\tfrac{7}{960}\, f_4 + \tfrac{7}{92160}\, f_0\, f_1\, f_3 + \tfrac{1}{55296}\, f_1^2\, f_2 - \tfrac{1}{31850496}\, f_0\, f_1^4\\
(\delta_2\, h)_3 =& -\tfrac{7}{138240}\, f_3\, f_2 - \tfrac{7}{69120}\, f_4\, f_1 - \tfrac{31}{96768}\, f_0\, f_5 + \tfrac{7}{13271040}\, f_0\, f_3\, f_1^2\\
&+\tfrac{1}{11943936}\, f_1^3\, f_2 - \tfrac{1}{11466178560}\, f_0\, f_1^5\\
(\delta_2\, h)_4 =& +\tfrac{31}{64512}\, f_6 - \tfrac{7}{13271040}\, f_3\, f_1\, f_2 - \tfrac{31}{9289728}\, f_0\, f_1\, f_5 - \tfrac{49}{88473600}\, f_0\, f_3^2\\
&-\tfrac{7}{13271040}\, f_1^2\, f_4 + \tfrac{7}{3822059520}\, f_0\, f_1^3\, f_3 + \tfrac{1}{4586471424}\, f_1^4\, f_2\\
&-\tfrac{1}{6604518850560}\, f_0\, f_1^6\\
(\delta_2\, h)_5 =& +\tfrac{31}{7741440}\, f_6\, f_1 + \tfrac{31}{23224320}\, f_2\, f_5 + \tfrac{127}{7372800}\, f_0\, f_7 + \tfrac{49}{55296000}\, f_3\, f_4\\
&-\tfrac{7}{3185049600}\, f_1^2\, f_2\, f_3 - \tfrac{49}{10616832000}\, f_0\, f_3^2\, f_1 - \tfrac{31}{2229534720}\, f_0\, f_1^2\, f_5\\
&-\tfrac{7}{4777574400}\, f_1^3\, f_4 + \tfrac{7}{1834588569600}\, f_0\, f_1^4\, f_3 + \tfrac{1}{2751882854400}\, f_1^5\, f_2\\
&-\tfrac{1}{5547795834470400}\, f_0\, f_1^7\\
(\delta_3\, h)_0 =& +\tfrac{1}{24}\, f_3 - \tfrac{1}{864}\, f_0\, f_1\, f_2 - \tfrac{7}{5760}\, f_3\, f_0^2 - \tfrac{1}{6912}\, f_1^3 + \tfrac{1}{497664}\, f_0^2\, f_1^3\\
(\delta_3\, h)_1 =& +\tfrac{7}{720}\, f_0\, f_4 + \tfrac{17}{5760}\, f_3\, f_1 + \tfrac{1}{864}\, f_2^2 - \tfrac{7}{138240}\, f_3\, f_0^2\, f_1 - \tfrac{1}{41472}\, f_0\, f_1^2\, f_2\\
&-\tfrac{1}{497664}\, f_1^4 + \tfrac{1}{47775744}\, f_0^2\, f_1^4\\
(\delta_3\, h)_2 =& -\tfrac{7}{576}\, f_5 + \tfrac{7}{34560}\, f_0\, f_4\, f_1 + \tfrac{7}{69120}\, f_0\, f_3\, f_2 + \tfrac{1}{41472}\, f_1\, f_2^2 + \tfrac{1}{23040}\, f_3\, f_1^2\\
&+\tfrac{31}{96768}\, f_0^2\, f_5 - \tfrac{7}{13271040}\, f_3\, f_0^2\, f_1^2 - \tfrac{1}{5971968}\, f_0\, f_1^3\, f_2 - \tfrac{1}{95551488}\, f_1^5\\
&+\tfrac{1}{11466178560}\, f_0^2\, f_1^5\\
(\delta_3\, h)_3 =& W - \tfrac{31}{24192}\, f_0\, f_6 - \tfrac{7}{51840}\, f_4\, f_2 - \tfrac{5}{18144}\, f_1\, f_5 - \tfrac{7}{138240}\, f_3^2\\
&+\tfrac{31}{6967296}\, f_1\, f_0^2\, f_5 + \tfrac{7}{4976640}\, f_0\, f_1^2\, f_4 + \tfrac{1}{5971968}\, f_1^2\, f_2^2 + \tfrac{49}{66355200}\, f_0^2\, f_3^2\\
&+\tfrac{31}{119439360}\, f_1^3\, f_3 + \tfrac{7}{4976640}\, f_0\, f_3\, f_1\, f_2\\
&-\tfrac{7}{2866544640}\, f_0^2\, f_1^3\, f_3 - \tfrac{1}{1719926784}\, f_0\, f_1^4\, f_2 - \tfrac{1}{34398535680}\, f_1^6\\
&+\tfrac{1}{4953389137920}\, f_0^2\, f_1^6.
\end{aligned}
$$

10.6.2 Free-β case

$$
\begin{aligned}
(\delta_1\, h)_0 =& -f_1 \beta_1\\
(\delta_1\, h)_1 =& -2\, f_2 \beta_2 + \tfrac{1}{2}\, f_1^2 \beta_1^2\\
(\delta_1\, h)_2 =& -3\, f_3 \beta_3 + f_1\, f_2 \beta_1 \beta_2 - \tfrac{1}{12}\, f_1^3 \beta_1^3\\
(\delta_1\, h)_3 =& -4\, f_4 \beta_4 - 4\, f_4 \beta_4 + f_1\, f_3 \beta_1 \beta_3 + \tfrac{1}{3}\, f_2^2 \beta_2^2! - \tfrac{1}{6}\, f_1^2\, f_2 \beta_1^2 \beta_2 + \tfrac{1}{144}\, f_1^4 \beta_1^4\\
(\delta_1\, h)_4 =& -5\, f_5 \beta_5 + f_1\, f_4 \beta_1 \beta_4 + \tfrac{1}{2}\, f_2\, f_3 \beta_2 \beta_3 - \tfrac{1}{8}\, f_1^2\, f_3 \beta_1^2 \beta_3 - \tfrac{1}{12}\, f_1\, f_2^2 \beta_1 \beta_2^2\\
&+\tfrac{1}{72}\, f_1^3\, f_2 \beta_1^3 \beta_2 - \tfrac{1}{2880}\, f_1^5 \beta_1^5
\end{aligned}
$$

$$
\begin{aligned}
(\delta_1 h)_5 = & -6 f_6\beta_6 + f_1 f_5\beta_1\beta_5 + \tfrac{2}{5} f_2 f_4\beta_2\beta_4 + \tfrac{3}{20} f_3^2\beta_3^2 - \tfrac{1}{10} f_1^2 f_4\beta_1^2\beta_4 \\
& -\tfrac{1}{90} f_2^3\beta_2^3 - \tfrac{1}{10} f_1 f_2 f_3\beta_1\beta_2\beta_3 + \tfrac{1}{120} f_1^3 f_3\beta_1^3\beta_3 + \tfrac{1}{120} f_1^2 f_2^2\beta_1^2\beta_2^2 \\
& -\tfrac{1}{1440} f_1^4 f_2\beta_1^4\beta_2 + \tfrac{1}{86400} f_1^6\beta_1^6 \\
(\delta_2 h)_0 = & -f_2\beta_1 + f_0 f_2\beta_2 - \tfrac{1}{4} f_0 f_1^2\beta_1^2 \\
(\delta_2 h)_1 = & -3 f_3\beta_2 + 3 f_0 f_3\beta_3 + f_1 f_2\beta_1^2 - f_0 f_1 f_2\beta_1\beta_2 + \tfrac{1}{12} f_0 f_1^3\beta_1^3 \\
(\delta_2 h)_2 = & -6 f_4\beta_3 + \tfrac{3}{2} f_1 f_3\beta_1\beta_2 + 6 f_0 f_4\beta_4 + f_2^2\beta_1\beta_2 - \tfrac{1}{2} f_0 f_2^2\beta_2^2 \\
& -\tfrac{1}{4} f_1^2 f_2\beta_1^3 - \tfrac{3}{2} f_0 f_1 f_3\beta_1\beta_3 + \tfrac{1}{4} f_0 f_1^2 f_2\beta_1^2\beta_2 - \tfrac{1}{96} f_0 f_1^4\beta_1^4 \\
(\delta_2 h)_3 = & -10 f_5\beta_4 + 2 f_1 f_4\beta_1\beta_3 + f_2 f_3\beta_1\beta_3 + 10 f_0 f_5\beta_5 + f_2 f_3\beta_2^2 \\
& -2 f_0 f_1 f_4\beta_1\beta_4 - \tfrac{1}{4} f_1^2 f_3\beta_1^2\beta_2 - \tfrac{1}{3} f_1 f_2^2\beta_1^2\beta_2 - f_0 f_2 f_3\beta_2\beta_3 \\
& +\tfrac{1}{36} f_1^3 f_2\beta_1^4 + \tfrac{1}{4} f_0 f_1^2 f_3\beta_1^2\beta_3 + \tfrac{1}{6} f_0 f_1 f_2^2\beta_1\beta_2^2 \\
& -\tfrac{1}{36} f_0 f_1^3 f_2\beta_1^3\beta_2 + \tfrac{1}{1440} f_0 f_1^5\beta_1^5 \\
(\delta_3 h)_0 = & -f_3\beta_1 + \tfrac{1}{3} f_1 f_2\beta_2 + 2 f_0 f_3\beta_2 - \tfrac{1}{12} f_1^3\beta_1^2 - f_0^2 f_3\beta_3 \\
& -\tfrac{2}{3} f_0 f_1 f_2\beta_1^2 + \tfrac{1}{3} f_0^2 f_1 f_2\beta_1\beta_2 - \tfrac{1}{36} f_0^2 f_1^3\beta_1^3 \\
(\delta_3 h)_1 = & -4 f_4\beta_2 + f_1 f_3\beta_1^2 + f_1 f_3\beta_3 + \tfrac{2}{3} f_2^2\beta_1^2 + 8 f_0 f_4\beta_3 - 4 f_0^2 f_4\beta_4 \\
& -\tfrac{1}{3} f_1^2 f_2\beta_1\beta_2 - \tfrac{4}{3} f_0 f_2^2\beta_1\beta_2 - 2 f_0 f_1 f_3\beta_1\beta_2 + \tfrac{1}{36} f_1^4\beta_1^3 \\
& +\tfrac{1}{3} f_0^2 f_2^2\beta_2^2 + \tfrac{1}{3} f_0 f_1^2 f_2\beta_1^3 + f_0^2 f_1 f_3\beta_1\beta_3 - \tfrac{1}{6} f_0^2 f_1^2 f_2\beta_1^2\beta_2 \\
& +\tfrac{1}{144} f_0^2 f_1^4\beta_1^4 \\
(\delta_3 h)_2 = & -10 f_5\beta_3 + 2 f_1 f_4\beta_1\beta_2 + 20 f_0 f_5\beta_4 + 2 f_1 f_4\beta_4 + 3 f_2 f_3\beta_1\beta_2 \\
& -10 f_0^2 f_5\beta_5 - \tfrac{1}{4} f_1^2 f_3\beta_1^3 - \tfrac{1}{3} f_1 f_2^2\beta_1^3 - \tfrac{1}{6} f_1 f_2^2\beta_2^2 - \tfrac{1}{2} f_1^2 f_3\beta_1\beta_3 \\
& -2 f_0 f_2 f_3\beta_2^2 - 2 f_0 f_2 f_3\beta_1\beta_3 - 4 f_0 f_1 f_4\beta_1\beta_3 + \tfrac{1}{12} f_1^3 f_2\beta_1^2\beta_2 \\
& +\tfrac{2}{3} f_0 f_1 f_2^2\beta_1^2\beta_2 + \tfrac{1}{2} f_0 f_1^2 f_3\beta_1^2\beta_2 + f_0^2 f_2 f_3\beta_2\beta_3 \\
& +2 f_0^2 f_1 f_4\beta_1\beta_4 - \tfrac{1}{288} f_1^5\beta_1^4 - \tfrac{1}{18} f_0 f_1^3 f_2\beta_1^4 - \tfrac{1}{6} f_0^2 f_1 f_2^2\beta_1\beta_2^2 \\
& -\tfrac{1}{4} f_0^2 f_1^2 f_3\beta_1^2\beta_3 + \tfrac{1}{36} f_0^2 f_1^3 f_2\beta_1^3\beta_2 - \tfrac{1}{1440} f_0^2 f_1^5\beta_1^5.
\end{aligned}
$$

11 Tables relative to the 4_1 knot

11.1 The original generators *Lo* and *Loo*

$$
\begin{aligned}
[\tfrac{p}{q}] \; &:= \cos(\pi\,\tfrac{p}{q}) \\
\hat{J}o_2 \; &= \; 4\,[\tfrac{0}{2}]; \quad \hat{J}o_4 = 26\,[\tfrac{0}{4}]; \quad \hat{J}o_6 = 60\,[\tfrac{0}{6}] + 56\,[\tfrac{2}{6}] \\
\hat{J}o_8 \; &= \; 186\,[\tfrac{0}{8}] + 90\,[\tfrac{2}{8}]; \quad \hat{J}o_{10} = 348\,[\tfrac{0}{10}] + 366\,[\tfrac{2}{10}] + 22\,[\tfrac{4}{10}]
\end{aligned}
$$

$$
\begin{aligned}
\hat{Jo}_{12} &= 650\,[\tfrac{0}{12}] + 748\,[\tfrac{2}{12}] + 624\,[\tfrac{4}{12}]\\
\hat{Jo}_{14} &= 1396\,[\tfrac{0}{14}] + 1854\,[\tfrac{2}{14}] + 1030\,[\tfrac{4}{14}] + 568\,[\tfrac{6}{14}]\\
\hat{Jo}_{16} &= 2776\,[\tfrac{0}{16}] + 3804\,[\tfrac{2}{16}] + 2816\,[\tfrac{4}{16}] + 1570\,[\tfrac{6}{16}]\\
\hat{Jo}_{18} &= 4862\,[\tfrac{0}{18}] + 8078\,[\tfrac{2}{18}] + 6550\,[\tfrac{4}{18}] + 4802\,[\tfrac{6}{18}] + 1472\,[\tfrac{8}{18}]\\
\hat{Jo}_{20} &= 9864\,[\tfrac{0}{20}] + 16588\,[\tfrac{2}{20}] + 14484\,[\tfrac{4}{20}] + 10242\,[\tfrac{6}{20}]\\
&\quad +4646\,[\tfrac{8}{20}]\\
\hat{Jo}_{22} &= 19238\,[\tfrac{0}{22}] + 34168\,[\tfrac{2}{22}] + 29144\,[\tfrac{4}{22}] + 23360\,[\tfrac{6}{22}]\\
&\quad +14032\,[\tfrac{8}{22}] + 4882\,[\tfrac{10}{22}]\\
\hat{Jo}_{24} &= 36622\,[\tfrac{0}{24}] + 68070\,[\tfrac{2}{24}] + 61092\,[\tfrac{4}{24}] + 49618\,[\tfrac{6}{24}]\\
&\quad +36436\,[\tfrac{8}{24}] + 18362\,[\tfrac{10}{24}]\\
\hat{Jo}_{26} &= 72910\,[\tfrac{0}{26}] + 136798\,[\tfrac{2}{26}] + 123574\,[\tfrac{4}{26}] + 105408\,[\tfrac{6}{26}]\\
&\quad +78140\,[\tfrac{8}{26}] + 49554\,[\tfrac{10}{26}] + 17140\,[\tfrac{12}{26}]\\
\hat{Jo}_{28} &= 142414\,[\tfrac{0}{28}] + 270968\,[\tfrac{2}{28}] + 250954\,[\tfrac{4}{28}] + 217464\,[\tfrac{6}{28}]\\
&\quad +171476\,[\tfrac{8}{28}] + 118824\,[\tfrac{10}{28}] + 62910\,[\tfrac{12}{28}]\\
\hat{Jo}_{30} &= 276046\,[\tfrac{0}{30}] + 536500\,[\tfrac{2}{30}] + 501662\,[\tfrac{4}{30}] + 444608\,[\tfrac{6}{30}]\\
&\quad +367512\,[\tfrac{8}{30}] + 275698\,[\tfrac{10}{30}] + 168622\,[\tfrac{12}{30}] + 57032\,[\tfrac{14}{30}]\\
\hat{Jo}_{32} &= 546414\,[\tfrac{0}{32}] + 1059780\,[\tfrac{2}{32}] + 998970\,[\tfrac{4}{32}] + 899322\,[\tfrac{6}{32}]\\
&\quad +761478\,[\tfrac{8}{32}] + 597972\,[\tfrac{10}{32}] + 413774\,[\tfrac{12}{32}] + 208304\,[\tfrac{14}{32}]\\
\hat{Jo}_{34} &= 1069006\,[\tfrac{0}{34}] + 2090050\,[\tfrac{2}{34}] + 1978918\,[\tfrac{4}{34}] + 1807392\,[\tfrac{6}{34}]\\
&\quad +1566682\,[\tfrac{8}{34}] + 1276434\,[\tfrac{10}{34}] + 946554\,[\tfrac{12}{34}] + 579596\,[\tfrac{14}{34}]\\
&\quad +192316\,[\tfrac{16}{34}]\\
\hat{Jo}_{36} &= 2088162\,[\tfrac{0}{36}] + 4103632\,[\tfrac{2}{36}] + 3916262\,[\tfrac{4}{36}] + 3606154\,[\tfrac{6}{36}]\\
&\quad +3192168\,[\tfrac{8}{36}] + 2678024\,[\tfrac{10}{36}] + 2087512\,[\tfrac{12}{36}] + 1424860\,[\tfrac{14}{36}]\\
&\quad +723470\,[\tfrac{16}{36}]\\
\hat{Jo}_{38} &= 4092062\,[\tfrac{0}{38}] + 8053558\,[\tfrac{2}{38}] + 7720542\,[\tfrac{4}{38}] + 7184188\,[\tfrac{6}{38}]\\
&\quad +6434868\,[\tfrac{8}{38}] + 5524442\,[\tfrac{10}{38}] + 4463534\,[\tfrac{12}{38}] + 3272004\,[\tfrac{14}{38}]\\
&\quad +2000440\,[\tfrac{16}{38}] + 677250\,[\tfrac{18}{38}]\\
\hat{Jo}_{40} &= 7996624\,[\tfrac{0}{40}] + 15773130\,[\tfrac{2}{40}] + 15189306\,[\tfrac{4}{40}]\\
&\quad +14233202\,[\tfrac{6}{40}] + 12919072\,[\tfrac{8}{40}] + 11281960\,[\tfrac{10}{40}]\\
&\quad +9387198\,[\tfrac{12}{40}] + 7241732\,[\tfrac{14}{40}] + 4922634\,[\tfrac{16}{40}] + 2500390\,[\tfrac{18}{40}]
\end{aligned}
$$

$$
\begin{aligned}
\overset{\wedge}{Joo}_{2} &= 4\,[\tfrac{0}{2}] \\
\overset{\wedge}{Joo}_{4} &= 12\,[\tfrac{0}{4}] \\
\overset{\wedge}{Joo}_{6} &= 20\,[\tfrac{0}{6}] + 16\,[\tfrac{2}{6}] \\
\overset{\wedge}{Joo}_{8} &= 44\,[\tfrac{0}{8}] + 24\,[\tfrac{2}{8}] \\
\overset{\wedge}{Joo}_{10} &= 68\,[\tfrac{0}{10}] + 72\,[\tfrac{2}{10}] + 8\,[\tfrac{4}{10}] \\
\overset{\wedge}{Joo}_{12} &= 108\,[\tfrac{0}{12}] + 128\,[\tfrac{2}{12}] + 96\,[\tfrac{4}{12}] \\
\overset{\wedge}{Joo}_{14} &= 196\,[\tfrac{0}{14}] + 264\,[\tfrac{2}{14}] + 152\,[\tfrac{4}{14}] + 80\,[\tfrac{6}{14}] \\
\overset{\wedge}{Joo}_{16} &= 340\,[\tfrac{0}{16}] + 480\,[\tfrac{2}{16}] + 352\,[\tfrac{4}{16}] + 200\,[\tfrac{6}{16}] \\
\overset{\wedge}{Joo}_{18} &= 540\,[\tfrac{0}{18}] + 904\,[\tfrac{2}{18}] + 728\,[\tfrac{4}{18}] + 520\,[\tfrac{6}{18}] + 160\,[\tfrac{8}{18}] \\
\overset{\wedge}{Joo}_{20} &= 980\,[\tfrac{0}{20}] + 1664\,[\tfrac{2}{20}] + 1440\,[\tfrac{4}{20}] + 1032\,[\tfrac{6}{20}] + 472\,[\tfrac{8}{20}] \\
\overset{\wedge}{Joo}_{22} &= 1740\,[\tfrac{0}{22}] + 3104\,[\tfrac{2}{22}] + 2656\,[\tfrac{4}{22}] + 2128\,[\tfrac{6}{22}] + 1280\,[\tfrac{8}{22}] \\
&\quad +440\,[\tfrac{10}{22}] \\
\overset{\wedge}{Joo}_{24} &= 3052\,[\tfrac{0}{24}] + 5688\,[\tfrac{2}{24}] + 5088\,[\tfrac{4}{24}] + 4136\,[\tfrac{6}{24}] + 3008\,[\tfrac{8}{24}] \\
&\quad +1528\,[\tfrac{10}{24}] \\
\overset{\wedge}{Joo}_{26} &= 5596\,[\tfrac{0}{26}] + 10520\,[\tfrac{2}{26}] + 9512\,[\tfrac{4}{26}] + 8112\,[\tfrac{6}{26}] + 6016\,[\tfrac{8}{26}] \\
&\quad +3816\,[\tfrac{10}{26}] + 1328\,[\tfrac{12}{26}] \\
\overset{\wedge}{Joo}_{28} &= 10156\,[\tfrac{0}{28}] + 19360\,[\tfrac{2}{28}] + 17912\,[\tfrac{4}{28}] + 15552\,[\tfrac{6}{28}] \\
&\quad +12256\,[\tfrac{8}{28}] + 8496\,[\tfrac{10}{28}] + 4488\,[\tfrac{12}{28}] \\
\overset{\wedge}{Joo}_{30} &= 18412\,[\tfrac{0}{30}] + 35792\,[\tfrac{2}{30}] + 33448\,[\tfrac{4}{30}] + 29632\,[\tfrac{6}{30}] \\
&\quad +24480\,[\tfrac{8}{30}] + 18344\,[\tfrac{10}{30}] + 11240\,[\tfrac{12}{30}] + 3808\,[\tfrac{14}{30}] \\
\overset{\wedge}{Joo}_{32} &= 34124\,[\tfrac{0}{32}] + 66240\,[\tfrac{2}{32}] + 62424\,[\tfrac{4}{32}] + 56232\,[\tfrac{6}{32}] \\
&\quad +47592\,[\tfrac{8}{32}] + 37392\,[\tfrac{10}{32}] + 25864\,[\tfrac{12}{32}] + 13024\,[\tfrac{14}{32}] \\
\overset{\wedge}{Joo}_{34} &= 62860\,[\tfrac{0}{34}] + 122936\,[\tfrac{2}{34}] + 116408\,[\tfrac{4}{34}] + 106320\,[\tfrac{6}{34}] \\
&\quad +92168\,[\tfrac{8}{34}] + 75096\,[\tfrac{10}{34}] + 55704\,[\tfrac{12}{34}] + 34096\,[\tfrac{14}{34}] \\
&\quad +11312\,[\tfrac{16}{34}] \\
\overset{\wedge}{Joo}_{36} &= 116012\,[\tfrac{0}{36}] + 228032\,[\tfrac{2}{36}] + 217576\,[\tfrac{4}{36}] + 200360\,[\tfrac{6}{36}] \\
&\quad +177312\,[\tfrac{8}{36}] + 148768\,[\tfrac{10}{36}] + 115904\,[\tfrac{12}{36}] + 79136\,[\tfrac{14}{36}] \\
&\quad +40168\,[\tfrac{16}{36}] \\
\overset{\wedge}{Joo}_{38} &= 215340\,[\tfrac{0}{38}] + 423848\,[\tfrac{2}{38}] + 406344\,[\tfrac{4}{38}] + 378128\,[\tfrac{6}{38}] \\
&\quad +338688\,[\tfrac{8}{38}] + 290776\,[\tfrac{10}{38}] + 234952\,[\tfrac{12}{38}] + 172224\,[\tfrac{14}{38}] \\
&\quad +105296\,[\tfrac{16}{38}] + 35640\,[\tfrac{18}{38}]
\end{aligned}
$$

$$
\begin{aligned}
\overset{\wedge}{DJo}_n &:= \overset{\wedge}{Jo}_n - \tfrac{1}{2}\ \overset{\wedge}{Joo}_n = n\,(\overset{\wedge}{Lo}_n - \overset{\wedge}{Loo}_n)\\
\ldots\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\overset{\wedge}{DJo}_2 &= 0\\
\overset{\wedge}{DJo}_4 &= 2\,[\tfrac{0}{4}]\\
\overset{\wedge}{DJo}_6 &= 8\,[\tfrac{2}{6}]\\
\overset{\wedge}{DJo}_8 &= 10\,[\tfrac{0}{8}] - 6\,[\tfrac{2}{8}]\\
\overset{\wedge}{DJo}_{10} &= 8\,[\tfrac{0}{10}] + 6\,[\tfrac{2}{10}] - 18\,[\tfrac{4}{10}]\\
\overset{\wedge}{DJo}_{12} &= 2\,[\tfrac{0}{12}] - 20\,[\tfrac{2}{12}] + 48\,[\tfrac{4}{12}]\\
\overset{\wedge}{DJo}_{14} &= 24\,[\tfrac{0}{14}] + 6\,[\tfrac{2}{14}] - 34\,[\tfrac{4}{14}] + 8\,[\tfrac{6}{14}]\\
\overset{\wedge}{DJo}_{16} &= 56\,[\tfrac{0}{16}] - 36\,[\tfrac{2}{16}] - 30\,[\tfrac{6}{16}]\\
\overset{\wedge}{DJo}_{18} &= 2\,[\tfrac{0}{18}] - 58\,[\tfrac{2}{18}] - 2\,[\tfrac{4}{18}] + 122\,[\tfrac{6}{18}] + 32\,[\tfrac{8}{18}]\\
\overset{\wedge}{DJo}_{20} &= 64\,[\tfrac{0}{20}] - 52\,[\tfrac{2}{20}] + 84\,[\tfrac{4}{20}] - 78\,[\tfrac{6}{20}] - 74\,[\tfrac{8}{20}]\\
\overset{\wedge}{DJo}_{22} &= 98\,[\tfrac{0}{22}] + 24\,[\tfrac{2}{22}] - 72\,[\tfrac{4}{22}] - 48\,[\tfrac{6}{22}] - 48\,[\tfrac{8}{22}] + 42\,[\tfrac{10}{22}]\\
\overset{\wedge}{DJo}_{24} &= -2\,[\tfrac{0}{24}] - 186\,[\tfrac{2}{24}] + 36\,[\tfrac{4}{24}] - 14\,[\tfrac{6}{24}] + 340\,[\tfrac{8}{24}] + 26\,[\tfrac{10}{24}]\\
\overset{\wedge}{DJo}_{26} &= 162\,[\tfrac{0}{26}] + 38\,[\tfrac{2}{26}] - 82\,[\tfrac{4}{26}] - 48\,[\tfrac{6}{26}] - 68\,[\tfrac{8}{26}] - 54\,[\tfrac{10}{26}]\\
&\quad -124\,[\tfrac{12}{26}]\\
\overset{\wedge}{DJo}_{28} &= 230\,[\tfrac{0}{28}] - 72\,[\tfrac{2}{28}] + 186\,[\tfrac{4}{28}] - 264\,[\tfrac{6}{28}] - 108\,[\tfrac{8}{28}]\\
&\quad -120\,[\tfrac{10}{28}] + 78\,[\tfrac{12}{28}]\\
\overset{\wedge}{DJo}_{30} &= -134\,[\tfrac{0}{30}] - 380\,[\tfrac{2}{30}] - 58\,[\tfrac{4}{30}] + 128\,[\tfrac{6}{30}] + 312\,[\tfrac{8}{30}]\\
&\quad +538\,[\tfrac{10}{30}] + 22\,[\tfrac{12}{30}] - 88\,[\tfrac{14}{30}]\\
\overset{\wedge}{DJo}_{32} &= 430\,[\tfrac{0}{32}] - 60\,[\tfrac{2}{32}] + 186\,[\tfrac{4}{32}] - 390\,[\tfrac{6}{32}] + 6\,[\tfrac{8}{32}] - 300\,[\tfrac{10}{32}]\\
&\quad -50\,[\tfrac{12}{32}] - 80\,[\tfrac{14}{32}]\\
\overset{\wedge}{DJo}_{34} &= 386\,[\tfrac{0}{34}] + 138\,[\tfrac{2}{34}] - 18\,[\tfrac{4}{34}] - 48\,[\tfrac{6}{34}] - 174\,[\tfrac{8}{34}] - 198\,[\tfrac{10}{34}]\\
&\quad -414\,[\tfrac{12}{34}] - 36\,[\tfrac{14}{34}] + 12\,[\tfrac{16}{34}]\\
\overset{\wedge}{DJo}_{36} &= -54\,[\tfrac{0}{36}] - 944\,[\tfrac{2}{36}] - 106\,[\tfrac{4}{36}] - 326\,[\tfrac{6}{36}] + 552\,[\tfrac{8}{36}]\\
&\quad +200\,[\tfrac{10}{36}] + 1240\,[\tfrac{12}{36}] + 412\,[\tfrac{14}{36}] + 446\,[\tfrac{16}{36}]\\
\overset{\wedge}{DJo}_{38} &= 602\,[\tfrac{0}{38}] + 446\,[\tfrac{2}{38}] + 6\,[\tfrac{4}{38}] - 244\,[\tfrac{6}{38}] - 204\,[\tfrac{8}{38}] - 302\,[\tfrac{10}{38}]\\
&\quad -554\,[\tfrac{12}{38}] - 252\,[\tfrac{14}{38}] - 184\,[\tfrac{16}{38}] + 90\,[\tfrac{18}{38}]\\
\overset{\wedge}{DJo}_{40} &= 704\,[\tfrac{0}{40}] - 150\,[\tfrac{2}{40}] + 666\,[\tfrac{4}{40}] - 558\,[\tfrac{6}{40}] + 672\,[\tfrac{8}{40}]\\
&\quad -600\,[\tfrac{10}{40}] - 162\,[\tfrac{12}{40}] - 828\,[\tfrac{14}{40}] - 726\,[\tfrac{16}{40}] - 570\,[\tfrac{18}{40}].
\end{aligned}
$$

11.2 The outer generators *Lu* and *Luu*

$$\overset{\frown}{\ell u}_3 = \frac{2}{3}\,\pi^2$$

$$\overset{\frown}{\ell u}_5 = \frac{47}{90}\,\pi^4$$

$$\overset{\frown}{\ell u}_7 = \frac{12361}{28350}\,\pi^6$$

$$\overset{\frown}{\ell u}_9 = \frac{10771487}{28576800}\,\pi^8$$

$$\overset{\frown}{\ell u}_{11} = \frac{23554574521}{70727580000}\,\pi^{10}$$

$$\overset{\frown}{\ell u}_{13} = \frac{108745677770927}{364105581840000}\,\pi^{12}$$

$$\overset{\frown}{\ell u}_{15} = \frac{941991271620333481}{3479028834481200000}\,\pi^{14}$$

$$\overset{\frown}{\ell u}_{17} = \frac{1405225117570189347436 7}{5677775057873318400000 0}\,\pi^{16}$$

$$\overset{\frown}{\ell u}_{19} = \frac{33823585328368123946674 5241}{14854762883913962929920 00000}\,\pi^{18}$$

$$\overset{\frown}{\ell u}_{21} = \frac{124807800799256412724191858178 07}{592705039068167120903808000000 00}\,\pi^{20}$$

$$\overset{\frown}{\ell u}_{23} = \frac{677464224865543063964283025725745801}{3463946059826089104698125094400000000}\,\pi^{22}$$

$$\overset{\frown}{\ell u}_{25} = \frac{52286450764941086752360153711155317617247}{286814733753600177869004757816320000000000}\,\pi^{24}$$

$$\overset{\frown}{\ell u}_{27} = \frac{5576709094295748822539682325837453212563351161}{32718390752941940290406717747896704000000000000}\,\pi^{26}$$

$$\overset{\frown}{\ell u}_{29} = \frac{80229625257347663179261323416371669245573858096 8687}{50212259920724936924881381593342113694720000000 00000}\,\pi^{28}$$

$$\overset{\frown}{\ell u}_{31} = \frac{1524781115516132193800161460746490584915121958041692 57321}{1015668487546463661648038146179327604759948800000000 0000000}\,\pi^{30}$$

$$\overset{\frown}{\ell u}_{33} = \frac{375944408413191413179416089612523682090657864171752279316 48127}{265991388866568275421681414026611747755373471334400000000 0000000}\,\pi^{32}$$

$$\overset{\frown}{\ell u}_{35} = \frac{118351414773276603212293650866715444757994669130460478291898544 40281}{887865956467161574943801475950128683420048878640660480000000000 0000000}\,\pi^{34}$$

$$\overset{\frown}{\ell u}_{37} = \frac{46907646740865188632656198851944801155136502232368723170832394131 23511567}{37253079801449165361492022327915499298938410850004832419840000000 000000000}\,\pi^{36}$$

$$\overset{\frown}{\ell uu}_2 = \frac{20}{81}\,\pi^2$$

$$\overset{\frown}{\ell uu}_4 = \frac{1219}{6561}\,\pi^4$$

$$\overset{\frown}{\ell uu}_6 = \frac{401353}{2657205}\,\pi^6$$

$$\overset{\frown}{\ell uu}_8 = \frac{36170973257}{281238577200}\,\pi^8$$

$$\overset{\frown}{\ell uu}_{10} = \frac{7690394022421}{68340974259600}\,\pi^{10}$$

$$\overset{\frown}{\ell uu}_{12} = \frac{671364005945401 3219}{669809888718339600 00}\,\pi^{12}$$

$$\overset{\frown}{\ell uu}_{14} = \frac{27080598984361034138281 1}{2995215671777347922904000 * Pi^{1}4}$$

$$\overset{\frown}{\ell uu}_{16} = \frac{15978614160083802139741148 6857}{19408997553117214540417920 00000}\,\pi^{16}$$

$$\overset{\frown}{\ell uu}_{18} = \frac{205842651548143071804675068 3489449}{272607134232552525105985853 56800000}\,\pi^{18}$$

$$\overset{\frown}{\ell uu}_{20} = \frac{84131570307269493011184643038 1192408987}{12077735169839497555491109612 39680000000}\,\pi^{20}$$

$$\widehat{\ell uu}_{22}=\frac{4846441341352181726344198698530254701229 0219}{7506296778045027938446770459139230451200000 00}\pi^{22}$$

$$\widehat{\ell uu}_{24}=\frac{262801525467520688318567177931229981418306153 4419657}{437426742473733906902927707668973364420717568 00000000}\pi^{24}$$

$$\widehat{\ell uu}_{26}=\frac{60844567261073718471236142418467451722253518312277829879}{108477077494980859630329300548561166404985557760000000000 0}\pi^{26}$$

$$\widehat{\ell uu}_{28}=\frac{93603597217612753243138615655392419573034242351802775833558 4419}{178245845519117719195329675465177057557014499002268160000000000000}\pi^{28}$$

$$\widehat{\ell uu}_{30}=\frac{54404189275775896728118322642154146046796828640171173653394056339 83}{110384411296434710448227752930575693980376224423031902976000000000000 0}\pi^{30}$$

$$\widehat{\ell uu}_{32}=\frac{934698272550163813181716127854037356437874429913136800167921387091723116 3657}{2016357601215648368128116515723911862323010614153503785708863488000000000000 000}\pi^{32}$$

$$\widehat{\ell uu}_{34}=\frac{2626109148916837130178692639592153052122950393809810251987365466545464729588 00889}{6011906026852326979569305917375060407367833696287414509977592665538560000000000 000 0}\pi^{34}.$$

11.3 The inner generators Li and Lii

$$\widehat{\ell\ i}_n := (\sqrt{3}\,\pi)^n\ \widehat{\ell i}^*_n \qquad \text{with} \qquad \widehat{\ell i}^*_n \in \mathbb{Q}^+ \qquad \forall n \in -\tfrac{1}{2}+\mathbb{N}$$

$$\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\widehat{\ell i}^*_{-1/2}=\frac{1}{2} \qquad \widehat{\ell i}^*_{1/2}=\frac{11}{108} \qquad \widehat{\ell i}^*_{3/2}=\frac{697}{34992} \qquad \widehat{\ell i}^*_{5/2}=\frac{724351}{141717600}$$

$$\widehat{\ell i}^*_{7/2} = \frac{278392949}{214277011200}$$

$$\widehat{\ell i}^*_{9/2} = \frac{244284791741}{728970392102400}$$

$$\widehat{\ell i}^*_{11/2} = \frac{1140363907117019}{12990252387264768000}$$

$$\widehat{\ell i}^*_{13/2} = \frac{212114205337147471}{9119157175859867136000}$$

$$\widehat{\ell i}^*_{15/2} = \frac{36736284422996813155 7}{5909213849957193904128 0000}$$

$$\widehat{\ell i}^*_{17/2} = \frac{449211928735297790783839 21}{268520586555904848197480448 00000}$$

$$\widehat{\ell i}^*_{19/2} = \frac{317434213056249557560214340 7}{70127812823436677081974711910 40000}$$

$$\widehat{\ell i}^*_{21/2} = \frac{69955029582443766280879140490 5733}{568603319153706921448359161640 7142400000}$$

$$\widehat{\ell i}^*_{23/2} = \frac{14222388631469863165732695954913158931}{4237231934333423978633172472546602516480 00000}$$

$$\widehat{\ell i}^*_{25/2} = \frac{5255000379400316520126835457783180207189}{57202631113501223711547828379379133972480 0000000}$$

$$\widehat{\ell i}^*_{27/2} = \frac{382316740742735395243571101350366537155843}{151638974842699607584430425049335958785228800000000}$$

$$\widehat{\ell i}^*_{29/2} = \frac{1616975399001217896007199158921134595564839756 0689}{233132868522150652042258816294562614423565854310400000000 00}$$

$$\widehat{\ell i}^*_{31/2} = \frac{119390469635156067915857712883546381438702433035719259}{62442307504972830642998601356335650647207878418497536000000000000}$$

$$\widehat{\ell i}^*_{33/2} = \frac{1116398659629170045249141261665722279335124967712466031771}{211417164750337009991064664472281245961316434749348957388800000000000 0}$$

$$\widehat{\ell i}^*_{35/2} = \frac{18640268802093894932827883966494225124248789786595330256830993}{127607308698179219159445514094866270747812639051453817474252800000000000000 0}$$

$$\widehat{\ell i}^*_{37/2} = \frac{246035809806427596769776498492549352551068621468202769029826362 91}{60798011431906342125367201938521746750600147981299588820925153280000000000000 00}$$

$$\widehat{\ell i}^*_{39/2} = \frac{1346006922185440944067965138463911390263357783573661737568466592369}{11990425211092485560028940347527593185248794401877171082248543272960000000000000 0}.$$

11.4 The exceptional generator *Le*

$$\overset{\frown}{\ell e}_{2n+1} \equiv 0$$

$$\overset{\frown}{\ell e}_{2n} \equiv \left(\tfrac{\pi}{4}\right)^{2n} \lambda^{-3n-2} \ \overset{\frown}{\ell e}{}^{*}_{2n} \quad \text{with} \quad \lambda := \log 2 \quad \text{and}$$

$$\overset{\frown}{\ell e}{}^{*}_{2n} \equiv \sum_{k=1}^{k=3n} \overset{\frown}{\ell e}{}^{*}_{2n,k} \ \lambda^k \ , \ \overset{\frown}{\ell e}{}^{*}_{2n,k} \in \mathbb{Q}$$

....

$$\overset{\frown}{\ell e}{}^{*}_{0} = \tfrac{1}{2}\lambda$$

$$\overset{\frown}{\ell e}{}^{*}_{2} = +\lambda - \tfrac{1}{6}\lambda^3$$

$$\overset{\frown}{\ell e}{}^{*}_{4} = +\tfrac{10}{3}\lambda + \tfrac{2}{3}\lambda^2 - \tfrac{2}{9}\lambda^3 - \tfrac{1}{9}\lambda^4 + \tfrac{1}{108}\lambda^5 + \tfrac{7}{540}\lambda^6$$

$$\begin{aligned}\overset{\frown}{\ell e}{}^{*}_{6} = {} & +\tfrac{112}{9}\lambda + \tfrac{224}{45}\lambda^2 + \tfrac{26}{135}\lambda^3 - \tfrac{52}{135}\lambda^4 - \tfrac{43}{405}\lambda^5 + \tfrac{34}{2025}\lambda^6 + \tfrac{331}{24300}\lambda^7 \\ & - \tfrac{7}{12150}\lambda^8 - \tfrac{62}{42525}\lambda^9\end{aligned}$$

$$\begin{aligned}\overset{\frown}{\ell e}{}^{*}_{8} = {} & +\tfrac{440}{9}\lambda + \tfrac{88}{3}\lambda^2 + \tfrac{5872}{945}\lambda^3 - \tfrac{176}{315}\lambda^4 - \tfrac{1802}{2835}\lambda^5 - \tfrac{101}{945}\lambda^6 + \tfrac{2801}{85050}\lambda^7 \\ & + \tfrac{76}{4725}\lambda^8 - \tfrac{17677}{14288400}\lambda^9 - \tfrac{8383}{4762800}\lambda^{10} + \tfrac{941}{23814000}\lambda^{11} + \tfrac{2159}{11907000}\lambda^{12}\end{aligned}$$

$$\begin{aligned}\overset{\frown}{\ell e}{}^{*}_{10} = {} & +\tfrac{16016}{81}\lambda + \tfrac{64064}{405}\lambda^2 + \tfrac{2312024}{42525}\lambda^3 + \tfrac{277024}{42525}\lambda^4 - \tfrac{262084}{127575}\lambda^5 - \tfrac{18416}{18225}\lambda^6 \\ & - \tfrac{8186}{91125}\lambda^7 + \tfrac{7864}{127575}\lambda^8 + \tfrac{904423}{45927000}\lambda^9 - \tfrac{220411}{80372250}\lambda^{10} - \tfrac{4359167}{1928934000}\lambda^{11} \\ & + \tfrac{235303}{2411167500}\lambda^{12} + \tfrac{754321}{3214890000}\lambda^{13} - \tfrac{1469}{482233500}\lambda^{14} - \tfrac{4526}{189448875}\lambda^{15}\end{aligned}$$

$$\begin{aligned}\overset{\frown}{\ell e}{}^{*}_{12} = {} & +\tfrac{198016}{243}\lambda + \tfrac{198016}{243}\lambda^2 + \tfrac{1369472}{3645}\lambda^3 + \tfrac{1637584}{18225}\lambda^4 + \tfrac{15535676}{4209975}\lambda^5 \\ & - \tfrac{20733676}{4209975}\lambda^6 - \tfrac{2758192}{1804275}\lambda^7 - \tfrac{1401608}{63149625}\lambda^8 + \tfrac{13879099}{126299250}\lambda^9 + \tfrac{198337}{8419950}\lambda^{10} \\ & - \tfrac{45253739}{7956852750}\lambda^{11} - \tfrac{474994117}{159137055000}\lambda^{12} + \tfrac{462208231}{1909644660000}\lambda^{13} \\ & + \tfrac{602881969}{1909644660000}\lambda^{14} - \tfrac{89414441}{10503045630000}\lambda^{15} - \tfrac{3387032899}{105030456300000}\lambda^{16} \\ & + \tfrac{48691583}{183803298525000}\lambda^{17} + \tfrac{977403607}{298680360103125}\lambda^{18}\end{aligned}$$

$$\begin{aligned}\overset{\frown}{\ell e}{}^{*}_{14} = {} & +\tfrac{826880}{243}\lambda + \tfrac{330752}{81}\lambda^2 + \tfrac{59576704}{25515}\lambda^3 + \tfrac{33079552}{42525}\lambda^4 + \tfrac{3920611264}{29469825}\lambda^5 \\ & - \tfrac{79270784}{11609325}\lambda^6 - \tfrac{11637846268}{1149323175}\lambda^7 - \tfrac{1380040216}{638512875}\lambda^8 + \tfrac{2690948354}{17239847625}\lambda^9 \\ & + \tfrac{153941804}{820945125}\lambda^{10} + \tfrac{6173415169}{241357866750}\lambda^{11} - \tfrac{1334139179}{120678933375}\lambda^{12} \\ & - \tfrac{28535702003}{7240736002500}\lambda^{13} + \tfrac{1985726927}{3620368001250}\lambda^{14} + \tfrac{2489824430263}{5734662913980000}\lambda^{15} \\ & - \tfrac{21102007}{907670610000}\lambda^{16} - \tfrac{11083211947}{247794076530000}\lambda^{17} + \tfrac{534608160781}{652317906465225000}\lambda^{18} \\ & + \tfrac{988383013399}{217439302155075000}\lambda^{19} - \tfrac{567191}{22187683893375}\lambda^{20} - \tfrac{357848408}{776568936268125}\lambda^{21}.\end{aligned}$$

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Brjuno conditions for linearization in presence of resonances

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Abstract. We present a new proof, under a slightly different (and more natural) arithmetic hypothesis, and using direct computations via power series expansions, of a holomorphic linearization result in presence of resonances originally proved by Rüssmann.

1 Introduction

We consider a germ of biholomorphism f of $\mathbb{C}^n$ at a fixed point p, which, up to translation, we may place at the origin O. One of the main questions in the study of local holomorphic dynamics (see [1, 2, 4], or [11, Chapter 1], for general surveys on this topic) is when f is *holomorphically linearizable*, *i.e.*, when there exists a local holomorphic change of coordinates such that f is conjugated to its linear part Λ.

A way to solve such a problem is to first look for a formal transformation φ solving

$$f \circ \varphi = \varphi \circ \Lambda,$$

i.e., to ask when f is *formally linearizable*, and then to check whether φ is convergent. Moreover, since up to linear changes of the coordinates we can always assume Λ to be in Jordan normal form, *i.e.*,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_2 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_n & \lambda_n \end{pmatrix},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ are not necessarily distincts, and $\varepsilon_j \in \{0, \varepsilon\}$ can be non-zero only if $\lambda_{j-1} = \lambda_j$, we can reduce ourselves to study such germs, and to search for φ *tangent to the identity*, that is, with linear part equal to the identity.

The answer to this question depends on the set of eigenvalues of $\mathrm{d}f_O$, usually called the *spectrum* of $\mathrm{d}f_O$. In fact, if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ the eigenvalues of $\mathrm{d}f_O$, then it may happen that there exists a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, with $|Q| \geq 2$, such that

$$\lambda^Q - \lambda_j := \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_j = 0 \tag{1.1}$$

for some $1 \leq j \leq n$; a relation of this kind is called a (*multiplicative*) *resonance* of f *relative to the* j*-th coordinate*, Q is called a *resonant multi-index relative to the* j*-th coordinate*, and we put

$$\mathrm{Res}_j(\lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

The elements of $\mathrm{Res}(\lambda) := \bigcup_{j=1}^n \mathrm{Res}_j(\lambda)$ are simply called *resonant* multi-indices. A *resonant monomial* is a monomial $z^Q := z_1^{q_1} \cdots z_n^{q_n}$ in the j-th coordinate with $Q \in \mathrm{Res}_j(\lambda)$.

Resonances are the formal obstruction to linearization. Indeed, we have the following classical result:

Theorem 1.1 (Poincaré, 1893 [7]; Dulac, 1904 [6]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O with linear part in Jordan normal form. Then there exists a formal transformation φ of $\mathbb{C}^n$, without constant term and tangent to the identity, conjugating f to a formal power series $g \in \mathbb{C}[[z_1, \ldots, z_n]]^n$ without constant term, with same linear part and containing only resonant monomials. Moreover, the resonant part of the formal change of coordinates φ can be chosen arbitrarily, but once this is done, φ and g are uniquely determined. In particular, if the spectrum of $\mathrm{d}f_O$ has no resonances, f is formally linearizable and the formal linearization is unique.*

A formal transformation g of $\mathbb{C}^n$, without constant term, and with linear part in Jordan normal form with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$, is called *in Poincaré-Dulac normal form* if it contains only resonant monomials with respect to $\lambda_1, \ldots, \lambda_n$.

If f is a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, a series g in Poincaré-Dulac normal form formally conjugated to f is called a *Poincaré-Dulac* (*formal*) *normal form of* f.

The problem with Poincaré-Dulac normal forms is that, usually, they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including finitely many resonant monomials only. This is indeed the case (see, *e.g.*, Reich [12]) when $\mathrm{d}f_O$ belongs to the so-called *Poincaré domain*, that is when $\mathrm{d}f_O$ is invertible and O is either *attracting*, *i.e.*, all the eigenvalues of $\mathrm{d}f_O$ have modulus less than 1, or *repelling*, *i.e.*, all the eigenvalues of $\mathrm{d}f_O$ have modulus greater than 1 (when $\mathrm{d}f_O$ is still invertible but does not belong to the Poincaré domain, we shall say that it belongs to the *Siegel domain*).

Even without resonances, the holomorphic linearization is not guaranteed. The best positive result is essentially due to Brjuno [5]. To describe such a result, let us introduce the following definitions:

Definition 1.2. For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $m \geq 2$ set

$$\omega_{\lambda_1,\ldots,\lambda_n}(m) = \min_{\substack{2\leq|Q|\leq m\\ 1\leq j\leq n}} |\lambda^Q - \lambda_j|. \tag{1.2}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\omega_f(m)$ for $\omega_{\lambda_1,\ldots,\lambda_n}(m)$.

It is clear that $\omega_f(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonances. It is also not difficult to prove that if f belongs to the Siegel domain then

$$\lim_{m\to+\infty} \omega_f(m) = 0,$$

which is the reason why, even without resonances, the formal linearization might be divergent.

Definition 1.3. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that λ *satisfies the Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu\geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{\lambda_1,\ldots,\lambda_n}(p_{\nu+1})} < \infty. \tag{1.3}$$

Brjuno's argument for vector fields, when adapted to the case of germs of biholomorphisms, yields the following result (see [8]).

Theorem 1.4 (Brjuno, 1971 [5]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, such that $\mathrm{d}f_O$ is diagonalizable. Assume moreover that the spectrum of $\mathrm{d}f_O$ has no resonances and satisfies the Brjuno condition. Then f is holomorphically linearizable.*

In the resonant case, one can still find formally linearizable germs, (see for example [9] and [10]), so two natural questions arise.

(Q1) *How many Poincaré-Dulac formal normal forms does a formally linearizable germ have*?

(Q2) *Is it possible to find arithmetic conditions on the eigenvalues of the spectrum of $\mathrm{d}f_O$ ensuring holomorphic linearizability of formally linearizable germs*?

Rüssmann gave answers to both questions in [13], an I.H.E.S. preprint which is no longer available, and that was finally published in [14]. The answer to the first question is the following (the statement is slightly different from the original one presented in [14] but perfectly equivalent):

Theorem 1.5 (Rüssmann, 2002 [14]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If f is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.*

To answer to the second question, Rüssmann introduced the following condition, that we shall call Rüssmann condition.

Definition 1.6. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ *satisfies the Rüssmann condition* if there exists a function $\Omega \colon \mathbb{N} \to \mathbb{R}$ such that:

(i) $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$;
(ii) $\sum_{k \geq 1} \frac{1}{k^2} \log \Omega(k) < +\infty$, and
(iii) $|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$ for all $j = 1, \ldots n$ and for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j.

Rüssmann proved the following generalization of Brjuno's Theorem 1.4 (the statement is slightly different from the original one presented in [14] but perfectly equivalent).

Theorem 1.7 (Rüssmann, 2002 [14]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the Rüssmann condition, then f is holomorphically linearizable.*

We refer to [14] for the original proof and we limit ourselves to briefly recall here the main ideas. To prove these results, Rüssmann first studies the process of Poincaré-Dulac formal normalization using a functional iterative approach, without assuming anything on the diagonalizability of $\mathrm{d}f_O$. With this functional technique he proves Theorem 1.5; then he constructs a formal iteration process converging to a zero of the operator $\mathcal{F}(\varphi) := f \circ \varphi - \varphi \circ \Lambda$ (where Λ is the linear part of f), and, assuming Λ diagonal, he gives estimates for each iteration step, proving that, under what we called the Rüssmann condition, the process converges to a holomorphic linearization.

In this paper, we shall first present a direct proof of Theorem 1.5 using power series expansions. Then we shall give a direct proof, using explicit computations with power series expansions and then proving convergence via majorant series, of an analogue of Theorem 1.7 under the following slightly different assumption, which is the natural generalization to the resonant case of the condition introduced by Brjuno.

Definition 1.8. Let $n \geq 2$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For $m \geq 2$ set

$$\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \mathrm{Res}_j(\lambda)}} \min_{1 \leq j \leq n} |\lambda^Q - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \dots, \lambda_n)$ relative to $1 \leq j \leq n$, *i.e.*, $\lambda^Q - \lambda_j = 0$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\widetilde{\omega}_f(m)$ for $\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)$.

Definition 1.9. Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. We say that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(p_{\nu+1})} < \infty.$$

We shall then prove:

Theorem 1.10. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, then f is holomorphically linearizable.*

We shall also show that Rüssmann condition implies the reduced Brjuno condition and so our result implies Theorem 1.7. The converse is known to be true in dimension 1, as proved by Rüssmann in [14], but is not known in higher dimension.

The structure of this paper is as follows. In the next section we shall discuss properties of formally linearizable germs, and we shall give our direct proof of Theorem 1.5. In Section 3 we shall prove Theorem 1.10 using majorant series. In the last section we shall discuss relations between Rüssmann condition and the reduced Brjuno condition.

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2 Formally linearizable germs

In general, a germ f can have several Poincaré-Dulac formal normal forms; however, we can say something on the shape of the formal conjugations between them. We have in fact the following result.

Proposition 2.1. *Let f and g be two germs of biholomorphism of $\mathbb{C}^n$ fixing the origin, with the same linear part Λ and in Poincaré-Dulac normal form. If there exists a formal transformation φ of $\mathbb{C}^n$, with no constant term and tangent to the identity, conjugating f and g, then φ contains only monomials that are resonant with respect to the eigenvalues of Λ.*

Proof. Since f and g are in Poincaré-Dulac normal form, Λ is in Jordan normal form. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of Λ. We shall prove that a formal solution $\varphi = I + \widehat{\varphi}$ of

$$f \circ \varphi = \varphi \circ g \tag{2.1}$$

contains only monomials that are resonant with respect to $\lambda_1, \dots, \lambda_n$. Using the standard multi-index notation, for each $j \in \{1, \dots, n\}$ we can write

$$f_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j f_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} z^Q,$$

$$g_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} g_{Q,j} z^Q,$$

and

$$\varphi_j(z) = z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\right) = z_j + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} \varphi_{Q,j} z^Q + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q \neq 1}} \varphi_{Q,j} z^Q,$$

where

$$N_j := \{Q \in \mathbb{Z}^n \mid |Q| \geq 1, q_j \geq -1, q_h \geq 0 \text{ for all } h \neq j\},$$

and $\varepsilon_j \in \{0, 1\}$ can be non-zero only if $\lambda_j = \lambda_{j-1}$. With these notations, the left-hand side of the j-th coordinate of (2.1) becomes

$$\begin{aligned}
&(f \circ \varphi)_j(z) \\
&= \lambda_j \varphi_j(z) + \varepsilon_j \varphi_{j-1}(z) + \varphi_j(z) \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} \prod_{k=1}^n \varphi_k(z)^{q_k} \\
&= \lambda_j z_j \big(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\big) + \varepsilon_j z_{j-1} \big(1 + \varphi_{j-1}^{\mathrm{res}}(z) + \varphi_{j-1}^{\neq\mathrm{res}}(z)\big) \\
&\quad + z_j \big(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\big) \sum_{\substack{Q \in N_j \\ \lambda^Q = \lambda_j}} f_{Q,j} z^Q \prod_{k=1}^n \big(1 + \varphi_k^{\mathrm{res}}(z) + \varphi_k^{\neq\mathrm{res}}(z)\big)^{q_k},
\end{aligned} \tag{2.2}$$

while the j-th coordinate of the right-hand side of (2.1) becomes

$$\begin{aligned}
&(\varphi\circ g)_j(z)\\
&=g_j(z)+g_j(z)\sum_{\substack{Q\in N_j\\ \lambda^Q=1}}\varphi_{Q,j}\prod_{k=1}^n g_k(z)^{q_k}+g_j(z)\sum_{\substack{Q\in N_j\\ \lambda^Q\neq1}}\varphi_{Q,j}\prod_{k=1}^n g_k(z)^{q_k}\\
&=\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\\
&\quad+\left(\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\right)\sum_{\substack{Q\in N_j\\ \lambda^Q=1}}\varphi_{Q,j}z^Q\prod_{k=1}^n\left(\lambda_k+\varepsilon_k\frac{z_{k-1}}{z_k}+g_k^{\mathrm{res}}(z)\right)^{q_k}\\
&\quad+\left(\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\right)\sum_{\substack{Q\in N_j\\ \lambda^Q\neq1}}\varphi_{Q,j}z^Q\prod_{k=1}^n\left(\lambda_k+\varepsilon_k\frac{z_{k-1}}{z_k}+g_k^{\mathrm{res}}(z)\right)^{q_k}.
\end{aligned}\tag{2.3}$$

Furthermore, notice that if P and Q are two multi-indices such that $\lambda^P=\lambda^Q=1$, then we have $\lambda^{\alpha P+\beta Q}=1$ for every $\alpha,\beta\in\mathbb{Z}$.

We want to prove that $\varphi_{Q,j}=0$ for each multi-index $Q\in N_j$ such that $\lambda^Q\neq 1$. Let us assume by contradiction that this is not true, and let $\widetilde{Q}$ be the first (with respect to the lexicographic order) multi-index in $N:=\bigcup_{j=1}^n N_j$ so that $\lambda^{\widetilde{Q}}\neq 1$ and $\varphi_{\widetilde{Q},j}\neq 0$. Let j be the minimal in $\{1,\dots,n\}$ such that $\widetilde{Q}\in N_j$, and let us compute the coefficient of the monomial $z^{\widetilde{Q}+e_j}$ in (2.2) and (2.3). In (2.2) we only have $\lambda_j\varphi_{\widetilde{Q},j}$ because, since $f-\Lambda$ is of second order and resonant, other contributions could come only from coefficients $\varphi_{P,k}$ with $|P|<|\widetilde{Q}|$ and $\lambda^P\neq 1$, but there are no such coefficients thanks to the minimality of $\widetilde{Q}$ and j. In (2.3) we can argue analogously, but we have also to take care of the monomials divisible by $\varepsilon_k^h(z_{k-1}/z_k)^h z^P$, with $\lambda^P=1$; in this last case, if $\varepsilon_k\neq 0$, we obtain a multi-index $P-he_k+he_{k-1}$, and again $\lambda^{P-he_k+he_{k-1}}=1$ because $\lambda_k=\lambda_{k-1}$. Then in (2.3) we only have $\lambda^{\widetilde{Q}+e_j}\varphi_{\widetilde{Q},j}$. Hence, we have

$$(\lambda^{\widetilde{Q}+e_j}-\lambda_j)\varphi_{\widetilde{Q},j}=0,$$

yielding

$$\varphi_{\widetilde{Q},j}=0,$$

because $\lambda^{\widetilde{Q}}\neq 1$ and $\lambda_j\neq 0$, and contradicting the hypothesis. □

Remark 2.2. It is clear from the proof that Proposition 2.1 holds also in the formal category, *i.e.*, for $f,g\in\mathbb{C}_O[\![z_1,\dots,z_n]\!]$ formal power series without constant terms in Poincaré-Dulac normal form.

We can now give a direct proof of Theorem 1.5, *i.e.*, that when a germ is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.

Theorem 2.3. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If f is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.*

Proof. Let Λ be the linear part of f. Up to linear conjugacy, we may assume that Λ is in Jordan normal form. If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ have no resonances, then there is nothing to prove. Let us then assume that we have resonances, and let us assume by contradiction that there is another Poincaré-Dulac formal normal form $g \not\equiv \Lambda$ associated to f. Since f is formally linearizable and it is formally conjugated to g, also g is formally linearizable. Thanks to Proposition 2.1, any formal linearization ψ of g tangent to the identity contains only monomials resonant with respect to $\lambda_1, \ldots, \lambda_n$; hence, writing $g = \Lambda + g^{\mathrm{res}}$ and $\psi = I + \psi^{\mathrm{res}}$, the conjugacy equation $g \circ \psi = \psi \circ \Lambda$ becomes

$$\begin{aligned}\Lambda + \Lambda\psi^{\mathrm{res}} + g^{\mathrm{res}} \circ (I + \psi^{\mathrm{res}}) &= (\Lambda + g^{\mathrm{res}}) \circ (I + \psi^{\mathrm{res}}) \\ &= (I + \psi^{\mathrm{res}}) \circ \Lambda \\ &= \Lambda + \psi^{\mathrm{res}} \circ \Lambda \\ &= \Lambda + \Lambda\psi^{\mathrm{res}},\end{aligned}$$

because $\psi^{\mathrm{res}} \circ \Lambda = \Lambda\psi^{\mathrm{res}}$. Hence there must be

$$g^{\mathrm{res}} \circ \psi \equiv 0,$$

and composing on the right with ψ^{-1} we get $g^{\mathrm{res}} \equiv 0$. □

Remark 2.4. As a consequence of the previous result, we get that *any formal normalization given by the Poincaré-Dulac procedure applied to a formally linerizable germ f is indeed a formal linearization of the germ.* In particular, we have *uniqueness of the Poincaré-Dulac normal form (which is linear and hence holomorphic), but not of the formal linearizations.* Hence *a formally linearizable germ f is formally linearizable via a formal transformation $\varphi = \mathrm{Id} + \widehat{\varphi}$ containing only non-resonant monomials.* In fact, thanks to the standard proof of Poincaré-Dulac Theorem (see [11, Theorem 1.3.25]), we can consider the formal normalization obtained with the Poincaré-Dulac procedure and imposing $\varphi_{Q,j} = 0$ for all Q and j such that $\lambda^Q = \lambda_j$; and this formal transformation φ, by Theorem 2.3, conjugates f to its linear part.

3 Convergence under the reduced Brjuno condition

Now we have all the ingredients needed to prove Theorem 1.10.

Theorem 3.1. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that* $\mathrm{d}f_O$ *is diagonalizable. If f is formally linearizable and the spectrum of* $\mathrm{d}f_O$ *satisfies the reduced Brjuno condition, then f is holomorphically linearizable.*

Proof. Up to linear changes of the coordinates, we may assume that the linear part Λ of f is diagonal, *i.e.*, $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$. From the conjugacy equation

$$f \circ \varphi = \varphi \circ \Lambda, \tag{3.1}$$

writing $f(z) = \Lambda z + \sum_{|L|\geq 2} f_L z^L$, and $\varphi(w) = w + \sum_{|Q|\geq 2} \varphi_Q w^Q$, where f_L and φ_Q belong to $\mathbb{C}^n$, we have that coefficients of φ have to verify

$$\sum_{|Q|\geq 2} A_Q \varphi_Q w^Q = \sum_{|L|\geq 2} f_L \left(\sum_{|M|\geq 1} \varphi_M w^M \right)^L, \tag{3.2}$$

where

$$A_Q = \lambda^Q I_n - \Lambda.$$

The matrices A_Q are not invertible only when $Q \in \bigcup_{j=1}^n \mathrm{Res}_j(\lambda)$, but, thanks Remark 2.4, we can set $\varphi_{Q,j} = 0$ for all $Q \in \mathrm{Res}_j(\lambda)$; hence we just have to consider $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, and, to prove the convergence of the formal conjugation φ in a neighbourhood of the origin, it suffices to show that

$$\sup_Q \frac{1}{|Q|} \log \|\varphi_Q\| < \infty, \tag{3.3}$$

for $|Q| \geq 2$ and $Q \notin \cap_{j=1}^n \mathrm{Res}_j(\lambda)$.

Since f is holomorphic in a neighbourhood of the origin, there exists a positive number ρ such that $\|f_L\| \leq \rho^{|L|}$ for $|L| \geq 2$. The functional equation (3.1) remains valid under the linear change of coordinates $f(z) \mapsto \sigma f(z/\sigma)$, $\varphi(w) \mapsto \sigma\varphi(w/\sigma)$ with $\sigma = \max\{1, \rho^2\}$. Therefore we may assume that

$$\forall |L| \geq 2 \qquad \|f_L\| \leq 1.$$

It follows from (3.2) that for any multi-index $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$ we have

$$\|\varphi_Q\| \leq \varepsilon_Q^{-1} \sum_{\substack{Q_1+\cdots+Q_\nu=Q \\ \nu\geq 2}} \|\varphi_{Q_1}\| \cdots \|\varphi_{Q_\nu}\|, \tag{3.4}$$

where
$$\varepsilon_Q = \min_{\substack{1\le j\le n\\ Q\notin \mathrm{Res}_j(\lambda)}} |\lambda^Q - \lambda_j|.$$
We can define, inductively, for $m \ge 2$
$$\alpha_m = \sum_{\substack{m_1+\cdots+m_\nu=j\\ \nu\ge 2}} \alpha_{m_1}\cdots\alpha_{m_\nu},$$
and
$$\delta_Q = \varepsilon_Q^{-1} \max_{\substack{Q_1+\cdots+Q_\nu=Q\\ \nu\ge 2}} \delta_{Q_1}\cdots\delta_{Q_\nu},$$
for $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \ge 2$, with $\alpha_1 = 1$ and $\delta_E = 1$, where E is any integer vector with $|E| = 1$. Then, by induction, we have that
$$\|\varphi_Q\| \le \alpha_{|Q|}\delta_Q,$$
for every $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \ge 2$. Therefore, to establish (3.3) it suffices to prove analogous estimates for α_m and δ_Q.

It is easy to estimate α_m. Let $\alpha = \sum_{m\ge 1}\alpha_m t^m$. We have
$$\alpha - t = \sum_{m\ge 2}\alpha_m t^m = \sum_{m\ge 2}\left(\sum_{h\ge 1}\alpha_h t^h\right)^m = \frac{\alpha^2}{1-\alpha}.$$
This equation has a unique holomorphic solution vanishing at zero
$$\alpha = \frac{t+1}{4}\left(1 - \sqrt{1 - \frac{8t}{(1+t)^2}}\right),$$
defined for $|t|$ small enough. Hence,
$$\sup_m \frac{1}{m}\log\alpha_m < \infty,$$
as we want.

To estimate δ_Q we have to take care of small divisors. First of all, for each multi-index $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \ge 2$ we can associate to δ_Q a decomposition of the form
$$\delta_Q = \varepsilon_{L_0}^{-1}\varepsilon_{L_1}^{-1}\cdots\varepsilon_{L_p}^{-1}, \tag{3.5}$$
where $L_0 = Q$, $|Q| > |L_1| \ge \cdots \ge |L_p| \ge 2$ and $L_j \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ for all $j = 1,\dots,p$ and $p \ge 1$. In fact, we choose a decomposition $Q =$

$Q_1+\cdots+Q_\nu$ such that the maximum in the expression of δ_Q is achieved; obviously, Q_j does not belong to $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ for all $j=1,\dots,\nu$. We can then express δ_Q in terms of $\varepsilon_{Q_j}^{-1}$ and $\delta_{Q'_j}$ with $|Q'_j|<|Q_j|$. Carrying on this process, we eventually arrive at a decomposition of the form (3.5). Furthermore, for each multi-index $Q\notin\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q|\geq 2$, we can choose an index i_Q so that

$$\varepsilon_Q=|\lambda^Q-\lambda_{i_Q}|.$$

The rest of the proof follows closely in [9, proof of Theorem 5.1]. For the benefit of the reader, we report it here.

For $m\geq 2$ and $1\leq j\leq n$, we can define

$$N_m^j(Q)$$

to be the number of factors ε_L^{-1} in the expression (3.5) of δ_Q, satisfying

$$\varepsilon_L<\theta\,\widetilde{\omega}_f(m),\quad\text{and}\quad i_L=j,$$

where $\widetilde{\omega}_f(m)$ is defined in Definition 1.8, and in this notation can be expressed as

$$\widetilde{\omega}_f(m)=\min_{\substack{2\leq|Q|\leq m\\ Q\notin\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)}}\varepsilon_Q,$$

and θ is the positive real number satisfying

$$4\theta=\min_{1\leq h\leq n}|\lambda_h|\leq 1.$$

The last inequality can always be satisfied by replacing f by f^{-1} if necessary. Moreover we also have $\widetilde{\omega}_f(m)\leq 2$.

Notice that $\widetilde{\omega}_f(m)$ is non-increasing with respect to m and under our assumptions $\widetilde{\omega}_f(m)$ tends to zero as m goes to infinity. The following is the key estimate.

Lemma 3.2. *For $m\geq 2$, $1\leq j\leq n$ and $Q\notin\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, we have*

$$N_m^j(Q)\leq\begin{cases}0, & \text{if } |Q|\leq m,\\[4pt] \dfrac{2|Q|}{m}-1, & \text{if } |Q|>m.\end{cases}$$

Proof. The proof is done by induction on $|Q|$. Since we fix m and j throughout the proof, we write N instead of N_m^j.

For $|Q| \leq m$,

$$\varepsilon_Q \geq \widetilde{\omega}_f(|Q|) \geq \widetilde{\omega}_f(m) > \theta\, \widetilde{\omega}_f(m),$$

hence $N(Q) = 0$.

Assume now that $|Q| > m$. Then $2|Q|/m - 1 \geq 1$. Write

$$\delta_Q = \varepsilon_Q^{-1} \delta_{Q_1} \cdots \delta_{Q_\nu}, \quad Q = Q_1 + \cdots + Q_\nu, \quad \nu \geq 2,$$

with $|Q| > |Q_1| \geq \cdots \geq |Q_\nu|$; note that $Q - Q_1$ does not belong to $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, otherwise the other Q_h's would be in $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$. We have to consider the following different cases.

Case 1: $\varepsilon_Q \geq \theta\, \widetilde{\omega}_f(m)$ and i_Q arbitrary, or $\varepsilon_Q < \theta\, \widetilde{\omega}_f(m)$ and $i_Q \neq j$. Then

$$N(Q) = N(Q_1) + \cdots + N(Q_\nu),$$

and applying the induction hypotheses to each term we get $N(Q) \leq (2|Q|/m) - 1$.

Case 2: $\varepsilon_Q < \theta\, \widetilde{\omega}_f(m)$ and $i_Q = j$. Then

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_\nu),$$

and there are three different subcases.

Case 2.1: $|Q_1| \leq m$. Then

$$N(Q) = 1 < \frac{2|Q|}{m} - 1,$$

as we want.

Case 2.2: $|Q_1| \geq |Q_2| > m$. Then there is ν' such that $2 \leq \nu' \leq \nu$ and $|Q_{\nu'}| > m \geq |Q_{\nu'+1}|$, and we have

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_{\nu'}) \leq 1 + \frac{2|Q|}{m} - \nu' \leq \frac{2|Q|}{m} - 1.$$

Case 2.3: $|Q_1| > m \geq |Q_2|$. Then

$$N(Q) = 1 + N(Q_1),$$

and there are again three different subcases.

Case 2.3.1: $i_{Q_1} \neq j$. Then $N(Q_1) = 0$ and we are done.

Case 2.3.2: $|Q_1| \leq |Q| - m$ and $i_{Q_1} = j$. Then

$$N(Q) \leq 1 + 2\,\frac{|Q| - m}{m} - 1 < \frac{2|Q|}{m} - 1.$$

Case 2.3.3: $|Q_1| > |Q| - m$ and $i_{Q_1} = j$. The crucial remark is that $\varepsilon_{Q_1}^{-1}$ gives no contribute to $N(Q_1)$, as shown in the next lemma.

Lemma 3.3. *If $Q > Q_1$ with respect to the lexicographic order, Q, Q_1 and $Q - Q_1$ are not in $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, $i_Q = i_{Q_1} = j$ and*

$$\varepsilon_Q < \theta\,\widetilde{\omega}_f(m) \quad \textit{and} \quad \varepsilon_{Q_1} < \theta\,\widetilde{\omega}_f(m),$$

then $|Q - Q_1| = |Q| - |Q_1| \geq m$.

Proof. Before we proceed with the proof, notice that the equality $|Q - Q_1| = |Q| - |Q_1|$ is obvious since $Q > Q_1$.

Since we are supposing $\varepsilon_{Q_1} = |\lambda^{Q_1} - \lambda_j| < \theta\,\widetilde{\omega}_f(m)$, we have

$$|\lambda^{Q_1}| > |\lambda_j| - \theta\,\widetilde{\omega}_f(m) \geq 4\theta - 2\theta = 2\theta.$$

Let us suppose by contradiction $|Q - Q_1| = |Q| - |Q_1| < m$. By assumption, it follows that

$$\begin{aligned} 2\theta\,\widetilde{\omega}_f(m) &> \varepsilon_Q + \varepsilon_{Q_1} \\ &= |\lambda^Q - \lambda_j| + |\lambda^{Q_1} - \lambda_j| \\ &\geq |\lambda^Q - \lambda^{Q_1}| \\ &\geq |\lambda^{Q_1}|\,|\lambda^{Q-Q_1} - 1| \\ &\geq 2\theta\,\widetilde{\omega}_f(|Q - Q_1| + 1) \\ &\geq 2\theta\,\widetilde{\omega}_f(m), \end{aligned}$$

which is impossible. □

Using Lemma 3.3, Case 1 applies to δ_{Q_1} and we have

$$N(Q) = 1 + N(Q_{1_1}) + \cdots + N(Q_{1_{\nu_1}}),$$

where $|Q| > |Q_1| > |Q_{1_1}| \geq \cdots \geq |Q_{1_{\nu_1}}|$ and $Q_1 = Q_{1_1} + \cdots + Q_{1_{\nu_1}}$. We can do the analysis of Case 2 again for this decomposition, and we finish unless we run into Case 2.3.2 again. However, this loop cannot happen more than $m + 1$ times and we have to finally run into a different case. This completes the induction and the proof of Lemma 3.2. □

Since the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, there exists a strictly increasing sequence $\{p_\nu\}_{\nu\geq 0}$ of integers with $p_0 = 1$ and such that

$$\sum_{\nu\geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_f(p_{\nu+1})} < \infty. \tag{3.6}$$

We have to estimate

$$\frac{1}{|Q|} \log \delta_Q = \sum_{j=0}^{p} \frac{1}{|Q|} \log \varepsilon_{L_j}^{-1}, \quad Q \notin \bigcap_{j=1}^{n} \mathrm{Res}_j(\lambda).$$

By Lemma 3.2,

$$\begin{aligned}\operatorname{card}\left\{0\le j\le p : \theta\,\widetilde{\omega}_f(p_{\nu+1})\le\varepsilon_{L_j}<\theta\,\widetilde{\omega}_f(p_\nu)\right\}&\le N^1_{p_\nu}(Q)+\cdots N^n_{p_\nu}(Q)\\&\le\frac{2n|Q|}{p_\nu}\end{aligned}$$

for $\nu \geq 1$. It is also easy to see from the definition of δ_Q that the number of factors $\varepsilon_{L_j}^{-1}$ is bounded by $2|Q|-1$. In particular,

$$\operatorname{card}\left\{0 \le j \le p : \theta\,\widetilde{\omega}_f(p_1) \le \varepsilon_{L_j}\right\} \le 2n|Q| = \frac{2n|Q|}{p_0}.$$

Then,

$$\begin{aligned}\frac{1}{|Q|}\log\delta_Q &\le 2n\sum_{\nu\ge 0}\frac{1}{p_\nu}\log\frac{1}{\theta\,\widetilde{\omega}_f(p_{\nu+1})}\\&=2n\left(\sum_{\nu\ge 0}\frac{1}{p_\nu}\log\frac{1}{\widetilde{\omega}_f(p_{\nu+1})}+\log\frac{1}{\theta}\sum_{\nu\ge 0}\frac{1}{p_\nu}\right).\end{aligned}\tag{3.7}$$

Since $\widetilde{\omega}_f(m)$ tends to zero monotonically as m goes to infinity, we can choose some $\overline{m}$ such that $1 > \widetilde{\omega}_f(m)$ for all $m > \overline{m}$, and we get

$$\sum_{\nu\ge\nu_0}\frac{1}{p_\nu}\le\frac{1}{\log(1/\widetilde{\omega}_f(\overline{m}))}\sum_{\nu\ge\nu_0}\frac{1}{p_\nu}\log\frac{1}{\widetilde{\omega}_f(p_{\nu+1})},$$

where ν_0 verifies the inequalities $p_{\nu_0-1} \le \overline{m} < p_{\nu_0}$. Thus both series in parentheses in (3.7) converge thanks to (3.6). Therefore

$$\sup_Q\frac{1}{|Q|}\log\delta_Q<\infty$$

and this concludes the proof. □

When there are no resonances, we obtain Brjuno's Theorem 1.4.

Remark 3.4. If the reduced Brjuno condition is not satisfied, then there are formally linearizable germs that are not holomorphically linearizable. A first example is the following: let us consider the following germ of biholomorphism f of $(\mathbb{C}^2, O)$:

$$\begin{aligned}f_1(z,w)&=\lambda z+z^2,\\f_2(z,w)&=w,\end{aligned}\tag{3.8}$$

with $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, not a Brjuno number. We are in presence of resonances because $\mathrm{Res}_1(\lambda, 1) = \{P \in \mathbb{N}^2 \mid P = (1, p), p \geq 1\}$ and $\mathrm{Res}_2(\lambda, 1) = \{P \in \mathbb{N}^2 \mid P = (0, p), p \geq 2\}$. It is easy to prove that f is formally linearizable, but not holomorphically linearizable, because otherwise the holomorphic function $\lambda z + z^2$ would be holomorphically linearizable contradicting Yoccoz's result [15].

A more general example is the following:

Example 3.5. Let $n \geq 2$, and let $\lambda_1, \ldots, \lambda_s \in \mathbb{C}^*$, be $1 \leq s < n$ complex non-resonant numbers such that

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_{\lambda_1,\ldots,\lambda_s}(m)} = +\infty. \tag{3.9}$$

Then it is possible to find (see *e.g.* [11, Theorem 1.5.1]) a germ f of biholomorphism of $\mathbb{C}^s$ fixing the origin, with $\mathrm{d}f_O = \mathrm{Diag}(\lambda_1, \ldots, \lambda_s)$, formally linearizable (since there are no resonances) but not holomorphically linearizable. It is also possible to find $\mu_1, \ldots, \mu_r \in \mathbb{C}^*$, with $r = n - s$, such that the n-tuple $\lambda = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ *has only level s resonances* (see [9], where this definition was first introduced, for details), *i.e.*, for $1 \leq j \leq s$ we have

$$\mathrm{Res}_j(\lambda) = \{P \in \mathbb{N}^n \,\big|\, |P| \geq 2,\ p_l = \delta_{jl} \text{ for } l = 1, \ldots, s, \text{ and } \mu_1^{p_{s+1}} \cdots \mu_r^{p_n} = 1\},$$

where δ_{jl} is the Kroenecker's delta, and for $s + 1 \leq h \leq n$ we have

$$\mathrm{Res}_h(\lambda) = \{P \in \mathbb{N}^n \,\big|\, |P| \geq 2,\ p_1 = \cdots = p_s = 0,\ \mu_1^{p_{s+1}} \cdots \mu_r^{p_n} = \mu_{h-s}\}.$$

Then any germ of biholomorphism F of $\mathbb{C}^n$ fixing the origin of the form

$$\begin{aligned} F_j(z, w) &= f_j(z) && \text{for } j = 1, \ldots, s, \\ F_h(z, w) &= \mu_{h-s} w_{h-s} + \widetilde{F}_h(z, w) && \text{for } h = s + 1, \ldots, n, \end{aligned}$$

with

$$\mathrm{ord}_z(\widetilde{F}_h) \geq 1,$$

for $h = s + 1, \ldots, n$, where $(z, w) = (z_1, \ldots, z_s, w_1, \ldots w_r)$ are local coordinates of $\mathbb{C}^n$ at the origin, is formally linearizable (see [9, Theorem 4.1]), but $\lambda = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$ does not satisfy the reduced Brjuno condition (because of (3.9)) and F is not holomorphically linearizable. In fact, if F were holomorphically linearizable via a linearization Φ, tangent to the identity, then $F \circ \Phi = \Phi \circ \mathrm{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$. Hence, for each $1 \leq j \leq s$, we would have

$$\begin{aligned} (F \circ \Phi)_j(z, w) &= \lambda_j \Phi_j(z, w) + \widetilde{f}_j(\Phi_1(z, w), \ldots, \Phi_s(z, w)) \\ &= (\Phi \circ \mathrm{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r))_j(z, w) \\ &= \Phi_j(\lambda_1 z_1, \ldots, \lambda_s z_s, \mu_1 w_1, \ldots, \mu_r w_r), \end{aligned}$$

yielding

$$(F \circ \Phi)_j(z, 0) = \Phi_j(\lambda_1 z_1, \dots, \lambda_s z_s, 0, \dots, 0),$$

and thus the holomorphic germ φ of $\mathbb{C}^s$ fixing the origin defined by $\varphi_j(z) = \Phi_j(z, 0)$ for $j = 1, \dots, s$, would coincide with the unique formal linearization of f, that would then be convergent contradicting the hypotheses.

4 Rüssmann condition vs. reduced Brjuno condition

Rüssmann proves that, in dimension 1, his condition is equivalent to Brjuno condition (see [14, Lemma 8.2]), and he also proves the following result.

Lemma 4.1 (Rüssmann, 2002 [14]). *Let $\Omega\colon \mathbb{N} \to (0, +\infty)$ be a monotone non decreasing function, and let $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$. Then*

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) \leq \sum_{k \geq 2^{q+1}} \frac{1}{k^2} \log \Omega(k).$$

We have the following relation between the Rüssmann and the reduced Brjuno condition.

Lemma 4.2. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. If λ satisfies Rüssmann condition, then it also satisfies the reduced Brjuno condition.*

Proof. The function $\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)$ defined in Definition 1.8 satisfies

$$\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)^{-1} \leq \widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m+1)^{-1}$$

for all $m \in \mathbb{N}$, and

$$|\lambda^Q - \lambda_j| \geq \widetilde{\omega}_{\lambda_1,\dots\lambda_n}(|Q|)$$

for each $j = 1, \dots, n$ and each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j. Furthermore, by its definition, it is clear that any other function $\Omega\colon \mathbb{N} \to \mathbb{R}$ such that $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$, and satisfying, for any $j = 1, \dots n$,

$$|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$$

for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j, is such that

$$\frac{1}{\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)} \leq \Omega(m)$$

for all $m \in \mathbb{N}$. Hence

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(p_{\nu+1})} < \sum_{\nu \geq 0} \frac{1}{p_\nu} \log \Omega(p_{\nu+1})$$

for any strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$. Since λ satisfies Rüssmann condition, thanks to Lemma 4.1, there exists a function Ω as above such that

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) < +\infty,$$

with $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$, and we are done. □

We do not know whether the Rüssmann condition is equivalent to the reduced Brjuno condition in the multi-dimensional case. As we said, Rüssmann is able to prove that this is true in dimension one, but to do so he strongly uses the one-dimensional characterization of these conditions via continued fraction.

Added in proofs. In: J. RAISSY, *Holomorphic linearization of commuting germs of holomorphic maps*, arXiv:1005.3434v1, it is proved that the Rüssmann condition and the reduced Brjuno condition are indeed equivalent.

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Noncommutative symmetric functions and combinatorial Hopf algebras

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Abstract. We present on a few examples a class of algebras which are increasingly popular in Combinatorics, and tend to permeate other fields as well. In particular, some of these algebras have connections with mould calculus and resurgence theory. They can be approached in many different ways. Here, they will be regarded as generalizations of the algebra of symmetric functions.

1 Introduction

The theory of symmetric functions is a highly developed subject. One may say that it is as old as algebra itself, its first manifestation being the relations between the roots and the coefficients of a polynomial. However, the theory is not limited to the study of algebraic equations, and the nineteenth century witnessed many contributions motivated by algebraic geometry, elimination theory, interpolation problems, invariant theory or combinatorics. With the introduction of group representations by Frobenius and Schur at the beginning of the twentieth century, symmetric functions became a basic tool for character calculations, and nowadays, deformations of the classical families of symmetric functions, related to the deformations of the classical groups known as quantum groups, are a quite active area of research, and find applications in various fields (see [28] for a relatively recent sample, and [52] for the state of the art in 1995).

To better understand the algebra of symmetric functions, several algebras closely related to it have been introduced over the years. Among them, one finds the so called *combinatorial Hopf algebras*[1], whose first example is Gessel's algebra of quasi-symmetric functions [37], soon followed by Noncommutative symmetric functions [36], and then by alge-

[1] So far, there is no general agreement on the precise definition of these objects. It has been proposed to define them as graded connected bialgebras endowed with a character [2], which in my opinion is not enough restrictive. Another approach related to operads [51] leads to a precise description of an interesting subclass.

bras based on permutations, tableaux, trees, and various combinatorial objects [11, 16, 49, 50, 54, 57, 61].

It is indeed a common feature of these algebras to have bases labeled by combinatorial structures (bases of symmetric functions are labeled by integer partitions, whose combinatorics is highly non-trivial). In at least one of these bases, the structure constants of the algebra are non-negative integers, whose understanding is a basic issue.

It is thus reasonable to call such algebras "combinatorial". It remains to explain the "Hopf" part. This comes from the fact that symmetric functions have a natural Hopf algebra structure (implicitely exploited long before Hopf algebras were formally defined, see, *e.g.*, [33]), which can be naturally extended to the other algebras.

The same algebras can however be introduced with quite different motivations. Among them, the theory of operads [12, 42, 48] is a rich source of examples, the best known one being the Loday-Ronco algebra of planar binary trees [49]. The Hopf-algebraic interpretation of renormalization in quantum field theory initiated by Kreimer [45] leads to various algebras of trees, the most famous one being the Connes-Kreimer algebra [15]. Such algebras of trees had been previously considered for the formal [34] or numerical [35] solution of differential equations. Several examples occur in representation theory. Zelevinsky [71] has classified the examples coming from an inductive sequence of finite groups, which are commutative-cocommutative (and self-dual). Some noncommutative examples are known to be related to non-semisimple algebras (degenerate Hecke algebras) [5, 39, 47].

Recently, it has been realized that this theory was also related to several aspects of Ecalle's works. For example, mould calculus is well-suited to describe the relations between families of generators of the combinatorial Hopf algebras, and can be used to build an operad [13, 14]. And more surprisingly, we shall see that noncommutative symmetric functions arise in resurgence theory, in the guise of alien operators [22, 23, 25, 55].

This article is structured as follows. We shall start with a reminder about symmetric functions, with emphasis on the Hopf algebra structure. Next, we shall present noncommutative symmetric functions in some detail. After that, we shall embark for a random walk through more complicated examples

All our algebras are over some field $\mathbb{K}$ of characteristic 0.

2 Symmetric functions

Symmetric "functions" are polynomials in an infinite set of indeterminates X, which are invariant under permutations of the variables. The basic example is provided by the so called *elementary symmetric functions*, which can be compactly described by a generating series[2]

$$\lambda_t(X) \text{ or } E(t;X) = \prod_{i\geq 1}(1+tx_i) = \sum_{n\geq 0} e_n(X)t^n. \tag{2.1}$$

More explicitely,

$$e_n = \sum_{i_1>i_2>\ldots>i_n} x_{i_1}x_{i_2}\cdots x_{i_n}. \tag{2.2}$$

The "fundamental theorem" of the theory of symmetric functions [52, (2.4) page 20] states that the $e_n(X)$ are algebraically independent, and the algebra of symmetric functions is

$$\mathrm{Sym}(X) = \mathbb{K}[e_1, e_2, \ldots], \tag{2.3}$$

naturally graded by the degree in X.

Now, let $Y = \{y_i | i \geq 1\}$ be a second set of indeterminates, and denote by $X+Y$ the disjoint union of X and Y. For $f \in \mathrm{Sym}$, set

$$\Delta f = f(X+Y). \tag{2.4}$$

It is clear that $\lambda_t(X+Y) = \lambda_t(X)\lambda_t(Y)$, so that

$$\Delta e_n = \sum_{i=0}^{n} e_i(X)e_{n-i}(Y), \tag{2.5}$$

which determines Δf for any f thanks to the "fundamental theorem". Identifying a product $u(x)v(Y)$ with the element $u\otimes v$ in the tensor product of algebras $\mathrm{Sym}\otimes\mathrm{Sym}$, we can interpret Δ as a *comultiplication*, *i.e.*, a linear map $\mathrm{Sym} \to \mathrm{Sym}\otimes\mathrm{Sym}$. If Z is a third set of variables, the obvious equality

$$f((X+Y)+Z) = f(X+(Y+Z)) \tag{2.6}$$

means that Δ is *coassociative*. The no less obvious fact that $\Delta(fg) = \Delta(f)\Delta(g)$ means that Δ is a homomorphism of algebras, so that Δ endows Sym with the structure of a *bialgebra*. Being graded and *connected* (this just means that the degree 0 component is one dimensional), it is automatically a Hopf algebra (it has an *antipode*), but we don't need to

[2] $\lambda_t(X)$ is Grothendieck's notation for the generating series of exterior powers in a λ-ring. And indeed, Sym is the free λ-ring on one generator.

resort to this argument: the antipode is the map $S : f(X) \mapsto f(-X)$, where symmetric functions of $-X$ are defined by

$$\sigma_t(-X) = \sigma_t(X)^{-1} = \lambda_{-t}(X), \tag{2.7}$$

so that $f(X - X + Y) = f(Y)$, as expected. In particular, $f(X - X) = f(\emptyset)$ is the constant term of f, which we may denote by $\epsilon(f)$. Denoting by $\mu : f \otimes g \mapsto fg$ the multiplication map, and by $u : 1_{\mathbb{K}} \mapsto 1_{\mathrm{Sym}}$ we can rewrite this as

$$\mu \circ (I \otimes S) \circ \Delta = u \circ \epsilon. \tag{2.8}$$

This is the usual definition of the antipode: the inverse of the identity map I for the *convolution* $\star$ of endomorphisms, $F \star G = \mu \circ (F \otimes G) \circ \Delta$, for which the projection onto the scalar component $u \circ \epsilon$ is the neutral element. The linear form ϵ is called the *co-unit*.

Thus, we see that at this point, the Hopf algebra formalism essentially amounts to giving learned names to trivial properties. It will nevertheless be quite useful in the sequel, especially when we shall come to noncommutative generalizations. Actually, the Hopf structure of Sym is a typical example of a Hopf algebra associated with a group. If we denote by G the multiplicative group of formal power series with constant term 1,

$$G = 1 + t\mathbb{K}[[t]] = \{a(t) = 1 + a_1 t + a_2 t^2 + \cdots\}, \tag{2.9}$$

we can interpret e_n as the coordinate function

$$e_n(a(t)) = a_n, \tag{2.10}$$

and Sym as the algebra of polynomial functions on G. With this interpretation,

$$e_n(x(t)y(t)) = \sum_{i=0}^{n} x_i y_{n-i} = \sum_{i=0}^{n} (e_i \otimes e_{n-i})(x(t) \otimes y(t)). \tag{2.11}$$

The standard coproduct for functions on a group (*cf.* [1]) turns a function $f(x)$ into a function of two variables $\Delta f(x, y) = f(xy)$, and the antipode is $Sf(x) = f(x^{-1})$. Thus, this is precisely what the coproduct of Sym does, although this structure has many other (and older) interpretations.

This is perhaps a good place to mention that we can also consider e_n as a coordinate function on the group of formal diffeomorphisms of $\mathbb{R}$ tangent to the identity

$$G_1 = \{A(t) = ta(t) | a(t) \in G\}. \tag{2.12}$$

This leads to a second Hopf algebra structure on Sym, which is known as the *Faà di Bruno algebra* (see, *e.g.*, [26]).

A graded bialgebra has a graded dual, the direct sum of the duals of its homogeneous components, endowed with the product dual to its coproduct, and with the coproduct dual to its product. On can show that Sym is *self-dual*, *i.e.*, that it is isomorphic to its graded dual. This is equivalent to the existence of a scalar product satisfying

$$\langle f \cdot g,\ h\rangle = \langle f \otimes g,\ \Delta h\rangle. \tag{2.13}$$

This scalar product has a very explicit description. To define it, we need some more bases of Sym.

We first have to introduce the *complete homogeneous symmetric function* h_n, which is defined as the sum of all monomials of degree n. On their generating series

$$\sigma_t(X) \text{ or } H(t; X) = \prod_{i\geq 1}(1-tx_i)^{-1} = \sum_{n\geq 0} h_n(X)t^n = \lambda_{-t}(X)^{-1}, \tag{2.14}$$

we see their relation to the antipode: $h_n(X) = (-1)^n e_n(-X) = (-1)^n S(e_n)$.

From the e_n or the h_n, we can build bases of Sym, which are labeled by unordered sequences of positive integers (integer partitions), usually displayed as nonincreasing sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r} \text{ and } h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}. \tag{2.15}$$

These are examples of *multiplicative bases*. But there is also an obvious (non multiplicative) basis, the *monomial symmetric functions* (sums over orbits)

$$m_\lambda = \Sigma x^\lambda = \sum_{\mu\in\mathfrak{S}(\lambda)} x^\mu, \tag{2.16}$$

where $\mathfrak{S}(\lambda)$ denotes the set of distinct permutations of λ. The dimension of the homogeneous component Sym is thus the number of partitions of n,

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101,$$
$$135, 176, 231, 297, 385, 490, 627, 792, 1002 \ldots$$

a highly non-trivial combinatorial sequence. Now, Hall's scalar product, which is defined by

$$\langle h_\lambda,\ m_\mu\rangle = \delta_{\lambda\mu} \tag{2.17}$$

satisfies (2.13). The relation between the basis h and m is neatly expressed by the identity

$$\sigma_1(XY) = \prod_{i,j\geq 1}(1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(X) h_\lambda(Y), \tag{2.18}$$

a *Cauchy type* identity, which is satisfied by any pair (u, v) of mutually adjoint bases:

$$\sigma_1(XY) = \sum_\lambda u_\lambda(X)v_\lambda(Y) \quad \text{iff} \quad \langle u_\lambda,\ v_\mu\rangle = \delta_{\lambda\mu}, \tag{2.19}$$

(see [52, (4.6) page 63]).

The original Cauchy identity (a classical exercise on determinants) expresses precisely this for the so-called *Schur functions*, of which a possible definition (not the original one) is

$$s_\lambda = \det(h_{\lambda_i+j-i}). \tag{2.20}$$

It reads [52, (4.8) page 64]

$$\sigma_1(XY) = \sum_\lambda s_\lambda(X)s_\lambda(Y) \tag{2.21}$$

which is equivalent to the fact that the Schur functions form an orthonormal basis for the Hall scalar product.

The importance of the Schur functions comes from the fact that they encode the irreducible characters of the symmetric groups. This fundamental result, due to Frobenius, as well as the character theory of finite groups in general, reads

$$\chi^\lambda_\mu = \langle s_\lambda,\ p_\mu\rangle \tag{2.22}$$

where χ^λ_μ denotes the value of the irreducible character χ^λ on an element of the conjugacy class μ, and the p_n are the *power sums*:

$$p_n(X) = \sum_{i\geq 1} x_i^n, \qquad p_\mu = p_{\mu_1}\cdots p_{\mu_r}. \tag{2.23}$$

For further reference, let us note that $p_n(X+Y) = p_n(X) + p_n(Y)$, so that the p_n are primitive elements of Sym, and that their relation to the other generators is given by the identity [52, (2.10) page 23]

$$\sigma_t(X) = \exp\left[\sum_{m\geq 1} p_m(X)\frac{t^m}{m}\right]. \tag{2.24}$$

3 Noncommutative symmetric functions

3.1 Basic definitions

The Hopf algebra **Sym** of noncommutative symmetric functions has a very simple definition: replace the complete symmetric functions h_n by non-commuting indeterminates S_n, and keep the coproduct formula

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k}. \tag{3.1}$$

That is, **Sym** is the free associative algebra $\mathbb{K}\langle S_1, S_2, \ldots\rangle$ over an infinite sequence S_n, with the grading $\deg S_n = n$ and the coproduct as above. It can be realized in terms of polynomials over an auxiliary set $A = \{a_i | i \geq 1\}$ of noncommuting variables, endowed with a total order:

$$\sigma_t(A) = \overrightarrow{\prod_{i\geq 1}}(1 - ta_i)^{-1} = \sum_{n\geq 0} S_n(A)t^n \quad (\mapsto h_n \text{ for } a_i \mapsto x_i), \tag{3.2}$$

so that

$$\lambda_t(A) = \overleftarrow{\prod_{1\leq i}}(1 + ta_i) = \sum_{n\geq 0} \Lambda_n(A)t^n \quad (\mapsto e_n \text{ for } a_i \mapsto x_i). \tag{3.3}$$

Equivalently,

$$S_n(A) = \sum_{i_1\leq i_2\leq\ldots\leq i_n} a_{i_1}a_{i_2}\cdots a_{i_n}, \tag{3.4}$$

$$\Lambda_n(A) = \sum_{i_1>i_2\ldots>i_n} a_{i_1}a_{i_2}\cdots a_{i_n}. \tag{3.5}$$

Then, the coproduct is again

$$\Delta F = F(A + B) \tag{3.6}$$

where $A + B$ is now interpreted as the *ordinal sum* of the ordered sets A and B (the disjoint union, ordered by $a < b$ for $a \in A$ and $b \in B$, so that this sum is not commutative) and where A commutes with B for the multiplication (so that the terms in A and B can be separated).

This structure has again an obvious interpretation in terms of the multiplicative group of formal power series over a noncommutative algebra[3]. It has also more exotic interpretations. For example, one can learn in [3] that it is $H_*(\Omega\Sigma\mathbb{C}P^\infty))$, the homology of loop space of the suspension of the infinite dimensional complex projective space.

Calling this algebra *Noncommutative Symmetric Functions* implies to look at it in a special way. In particular, to look for analogues of the

[3] Although such series do not form a group under composition, the noncommutative version of the Faà di Bruno algebra does exist, and has applications in quantum field theory [10].

classical families of symmetric functions, of the numerous operations (internal products, plethysms) existing on Sym, and of the various interpretations of Sym. For example, in representation theory, **Sym** is to the tower of 0-Hecke algebras what Sym is to the sequence of symmetric groups[4] [47].

As a Hopf algebra, **Sym** is not self-dual. This is clear from the definition of the coproduct, which is obviously cocommutative. Its (graded) dual is the commutative algebra QSym of *quasi-symmetric functions* [32, 36], about which a few words will be said in the sequel.

3.2 Generators and linear bases

From the generators S_n, we can form a linear basis

$$S^I = S_{i_1} S_{i_2} \cdots S_{i_r} \tag{3.7}$$

of the homogeneous component $\mathbf{Sym}_n$, parametrized by *compositions* of n, that is, finite ordered sequences $I = (i_1, \ldots, i_r)$ of positive integers summing to n. One often writes $I \vDash n$ to mean that I is a composition of n. The dimension of $\mathbf{Sym}_n$ is 2^{n-1} for $n \geq 1$.

Similarly, from the Λ_n we can build a basis Λ^I. We can also look for analogues of the power-sum symmetric functions. It is here that the nontrivial questions arise. Indeed, in the commutative case, the power-sum p_n is, up to a scalar factor, the unique primitive element of degree n (this is easily seen by writing an arbitrary primitive element as a polynomial in the p_k). Here, the primitive elements form a free Lie algebra (more on this later), and it is not immediately obvious to identify the ones which should deserve the name "noncommutative power sums".

However, we can at least give one example: the series σ_t being grouplike ($\Delta\sigma_t = \sigma_t \otimes \sigma_t$), its logarithm is primitive, and writing

$$\log \sigma_t = \sum_{n \geq 1} \Phi_n \frac{t^n}{n}, \tag{3.8}$$

we can reasonably interpret Φ_n as a noncommutative power-sum (of course, it is not $\sum_i a_i^n$, this element is not even in **Sym**).

Recall now from (3.4) and (3.5) that in terms of our auxiliary ordered alphabet $A = \{a_1, a_2, \ldots\}$ S_n is the sum of nondecreasing words, and Λ_n the sum of strictly decreasing words.

[4] With this difference that these algebras are not semi-simple. To be precise, $\mathbf{Sym}_n$ is $K_0(H_n(0))$ (Grothendieck group of projective modules).

Let us say that a word $w = a_{i_1} a_{i_2} \cdots a_{i_n}$ has a *descent* at k if $i_k > i_{k+1}$, and denote by $\mathrm{Des}(w)$ the set of such k (the *descent set* of w).

Thus, $S_n(A)$ is the sum of words of length n with no descent, and $\Lambda_n(A)$ the sum of words with descents at all possible places. Now, obviously,

$$S^I = S_{i_1} S_{i_2} \cdots S_{i_r} \tag{3.9}$$

is the sum of words of length $i_1 + i_2 + \cdots + i_r$ whose descent set is contained in

$$\{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots i_{r-1}\}. \tag{3.10}$$

Let us denote this set by $\mathrm{Des}(I)$ and call it the *descent set of the composition I*. Symmetrically, we call I the *descent composition*, and write $I = C(w)$, of any word of length n having $\mathrm{Des}(I)$ as descent set.

It seems now natural to introduce the *noncommutative ribbon Schur functions*[5]

$$R_I(A) = \sum_{C(w)=I} w \tag{3.11}$$

so that we have

$$S^I = \sum_{J \le I} R_J \tag{3.12}$$

where $J \le I$ is the *reverse refinement order*, which means that $\mathrm{Des}(J) \subseteq \mathrm{Des}(I)$.

The commutative ribbon Schur functions were investigated by McMahon before 1915 in connection with combinatorial problems. We shall now see that the noncommutative version has an interesting interpretation.

3.3 Solomon's descent algebras

Apart from the already mentioned relation with 0-Hecke algebras, which will not be covered here (see [47]), there is another noncommutative analogue of the relation between symmetric functions and characters of symmetric groups. The story goes back to a discovery of Louis Solomon [66]. His construction, which provides a noncommutative lift in the group algebra of Mackey's formula for a product of induced characters, is valid in general for finite Coxeter groups, but we shall only need the case of symmetric groups.

[5] Their commutative images are indeed the so-called skew Schur functions indexed by ribbon diagrams.

Let (W, S) be a Coxeter system. One says that $w \in W$ has a descent at $s \in S$ if w has a reduced word ending by s. For $W = \mathfrak{S}_n$ and $s_i = (i, i+1)$, this means that $w(i) > w(i+1)$, whence the terminology. In this case, we rather say that i is a descent of w. Let $\mathrm{Des}(w)$ denote the descent set of w, and for a subset $E \subseteq S$, set

$$D_E = \sum_{\mathrm{Des}(w)=E} w \quad \in \mathbb{Z}W. \tag{3.13}$$

Solomon has shown that the D_E span a $\mathbb{Z}$-subalgebra $\Sigma(W)$ of $\mathbb{Z}W$. Moreover

$$D_{E'} D_{E''} = \sum_E c^E_{E'E''} D_E \tag{3.14}$$

where the coefficients $c^E_{E'E''}$ are nonnegative integers.

In the case of $W = \mathfrak{S}_n$, we encode descent sets by compositions of n as explained above. If $E = \{d_1, \dots, d_{r-1}\}$, we set $d_0 = 0$, $d_r = n$ and $I = C(E) = (i_1, \dots, i_r)$, where $i_k = d_k - d_{k-1}$. From now on, we shall write D_I instead of D_E, and denote by Σ_n the descent algebra of $\mathfrak{S}_n$ (with coefficients in our ground field $\mathbb{K}$).

Thus, Σ_n has the same dimension as $\mathbf{Sym}_n$, and both have natural bases labelled by compositions of n. It is therefore tempting to look for a good isomorphism between them.

Remembering what has been said about descents, the map $\alpha : D_I \to R_I$ may appear as a natural choice for a correspondence $\Sigma_n \to \mathbf{Sym}_n$.

This choice is not only natural, it is *canonical*. Indeed, Solomon had proved that the structure constants of his descent algebra were the same as the decomposition coefficients of certain tensor products of representations of $\mathfrak{S}_n$. Precisely, if we set

$$B^I = \sum_{J \le I} D_I \ \in \Sigma_n \tag{3.15}$$

and

$$B^I B^J = \sum_K b^{IJ}_K B^K, \tag{3.16}$$

then, the Kronecker products of the characters β^I of $\mathfrak{S}_n$, induced by the trivial representations of the parabolic subgroups $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r}$, decompose as

$$\beta^I \beta^J = \sum_K b^{IJ}_K \beta^K. \tag{3.17}$$

Such products of induced characters can be calculated by Mackey's formula, whence the title of Solomon's paper [66].

So, why is the above correspondence canonical? This is because it is compatible with the Frobenius characteristic map, from $\mathfrak{S}_n$-characters to symmetric functions. Indeed, $\mathrm{ch}(\beta^I) = h_I$, the commutative image of S^I.

The Frobenius characteristic map allows one to define the *internal product* $*$ on symmetric functions, by setting $h_\lambda * h_\mu = \mathrm{ch}(\beta^\lambda \beta^\mu)$. We can now do the same on noncommutative symmetric functions, using the descent algebras instead of the character rings.

For technical reasons which will soon become clear, we want our correspondence to be an *anti-isomorphism.* We set

$$S^I * S^J = \sum_K b_K^{JI} S^K. \tag{3.18}$$

This is because we want to interpret permutations as endomorphisms of tensor algebras: if $f_\sigma(\mathbf{w}) = \mathbf{w}\sigma$, then $f_\sigma \circ f_\tau = f_{\tau\sigma}$ (see below).

3.4 Permutational operators on tensor spaces

Let V be a vector space over some field $\mathbb{K}$ of characteristic 0. Let $T(V)$ be its tensor algebra, and $L(V)$ the free Lie algebra generated by V. We denote by $L_n(V) = L(V) \cap V^{\otimes n}$ its homogeneous component of degree n. The group algebra $\mathbb{K}\mathfrak{S}_n$ acts on the right on $V^{\otimes n}$ by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{3.19}$$

This action commutes with the left action of $GL(V)$, and when $\dim V \geq n$, which we shall usually assume, these actions are the commutant of each other (Schur-Weyl duality).

Any $GL(V)$-equivariant projector $\Pi_n : V^{\otimes n} \to L_n(V)$ can therefore be regarded as an idempotent π_n of $\mathbb{K}\mathfrak{S}_n$: $\Pi_n(\mathbf{v}) = \mathbf{v} \cdot \pi_n$. By definition (*cf.* [64]), such an element is called a *Lie idempotent* whenever its image is $L_n(V)$. Then, a homogeneous element $P_n \in V^{\otimes n}$ is in $L_n(V)$ if and only if $P_n \pi_n = P_n$.

From now on, we fix a basis $A = \{a_1, a_2, \ldots\}$ of V. We identify $T(V)$ with the free associative algebra $\mathbb{K}\langle A\rangle$, and $L(V)$ with the free Lie algebra $L(A)$.

3.5 The Hausdorff series

Lie idempotents arise naturally in the investigation of the Hausdorff series

$$H(a_1, a_2, \ldots, a_N) = \log\left(e^{a_1} e^{a_2} \cdots e^{a_N}\right) = \sum_{n \geq 0} H_n(A) \tag{3.20}$$

which is known to be a *Lie series*, *i.e.*, each homogeneous component $H_n(A) \in L_n(A)$. This is known as the Baker-Campbell-Hausdorff (BCH) "formula". It follows immediately from the characterization of $L(A)$ as the space of primitive elements of the standard comultiplication of $\mathbb{K}\langle A\rangle$ (Friedrichs' criterion[6]).

Actually, the BCH formula is not a formula at all, only a (very important) property, which is the basis of the correspondence between Lie groups and Lie algebras. However, various applications (including real-world ones) require explicit calculation of the polynomials $H_n(A)$. This is where Lie idempotents come into play.

For small values of n and N, this can be done by hand, and one obtains for example

$$\begin{aligned}H_3(a_1, a_2, a_3) = {} & \frac{1}{12}a_1a_1a_2 + \frac{1}{12}a_1a_1a_3 - \frac{1}{6}a_1a_2a_1 + \frac{1}{12}a_1a_2a_2 \\ & + \frac{1}{3}a_1a_2a_3 - \frac{1}{6}a_1a_3a_1 + \frac{1}{12}a_1a_3a_3 + \frac{1}{12}a_2a_1a_1 \\ & - \frac{1}{6}a_2a_1a_2 - \frac{1}{6}a_2a_1a_3 + \frac{1}{12}a_2a_2a_1 + \frac{1}{12}a_2a_2a_3 \\ & - \frac{1}{6}a_2a_3a_1 - \frac{1}{6}a_2a_3a_2 + \frac{1}{12}a_2a_3a_3 + \frac{1}{12}a_3a_1a_1 \\ & - \frac{1}{6}a_3a_1a_2 - \frac{1}{6}a_3a_1a_3 + \frac{1}{3}a_3a_2a_1 + \frac{1}{12}a_3a_2a_2 \\ & - \frac{1}{6}a_3a_2a_3 + \frac{1}{12}a_3a_3a_1 + \frac{1}{12}a_3a_3a_2.\end{aligned}$$

Already, it might not be obvious, at first sight, that this is indeed a Lie polynomial, which can be rewritten in the form

$$\begin{aligned}H_3(a_1, a_2, a_3) = {} & \frac{1}{12}[a_1, [a_1, a_2]] + \frac{1}{12}[[a_1, a_2], a_2] + \frac{1}{12}[a_1, [a_1, a_3]] \\ & + \frac{1}{12}[[a_1, a_3], a_3] + \frac{1}{12}[a_2, [a_2, a_3]] + \frac{1}{12}[[a_2, a_3], a_3] \\ & + \frac{1}{6}[a_1, [a_2, a_3]] + \frac{1}{6}[[a_1, a_2], a_3].\end{aligned}$$

The problem of finding a systematic procedure for expressing the Hausdorff series as a linear combination of commutators was raised at the

[6] This criterion, which appears only as a footnote [29, page 203] in a text on quantum field theory, is for us, together with its converse (the Milnor-Moore theorem, due to Cartier) the "founding act" of combinatorial Hopf algebra theory, which has actually little to do with Hopf's original motivations.

Gelfand seminar in the 1940's, and Dynkin [20] came up with the following solution (also discovered independently by Specht [68] and Wever [69]):

Theorem 3.1.

$$\theta_n = \frac{1}{n}[\dots[[[1,2],3],\dots,],n] \tag{3.21}$$

is a Lie idempotent. Therefore, expanding H_n as a linear combination of words, and writing $H_n = H_n\theta_n$, gives the required expression.

In order to apply this recipe, we need a reasonably efficient way to find the expansion on words

$$H_n(A) = \sum_{w\in A^n} c_w w. \tag{3.22}$$

One can show that $H_n(A)$ is the image of the homogeneous component $E_n(A)$ of the product of exponentials

$$E(A) = e^{a_1}e^{a_2}\cdots e^{a_N} = \sum_{n\geq 0} E_n(A). \tag{3.23}$$

under an element of $\mathbb{K}\mathfrak{S}_n$

$$H_n(A) = E_n(A)\cdot\phi_n, \tag{3.24}$$

where

$$\phi_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}\frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\sigma \qquad (d(\sigma) = |\operatorname{Des}(\sigma)|). \tag{3.25}$$

Here, $\operatorname{Des}(\sigma) = \{i|\sigma(i) > \sigma(i+1)\}$ denotes the *descent set* of σ. This formula is due to Solomon [65] and independently to Bialynicki-Birula, Mielnik and Plebański [6]. It can be shown, although none of these properties is clearly apparent on the expression (3.25), that ϕ_n is a Lie idempotent. It has been rediscovered many times, and is also known as the (first) Eulerian idempotent.

We can now write the Hausdorff polynomials

$$H_n(A) = E_n(A)\phi_n\theta_n = \sum_{w\in A^n} c_w\cdot w\theta_n \tag{3.26}$$

as linear combinations of commutators $w\theta_n$. However, these commutators are far from being linearly independent, and one would be interested in an expansion of H_n on a *basis* of the free Lie algebra.

One way to achieve this is to use *Klyachko's basis* of the free Lie algebra. This little known basis is obtained from a third Lie idempotent, discovered by A. Klyachko [43], and originally introduced as the solution of a different problem.

This problem was the following. The character of $GL(V)$ on $L_n(V)$ is known (Witt has given formulas for the dimensions of its weight spaces), and its expression shows that as a $GL(V)$-module, $L_n(V)$ is isomorphic to the space $\Gamma_n(V)$, image of the idempotent

$$\gamma_n = \frac{1}{n}\sum_{k=0}^{n-1} \omega^k c^k, \tag{3.27}$$

where $c = (12\dots n)$ is an n-cycle of $\mathfrak{S}_n$ and ω a primitive nth root of unity. Klyachko's idempotent κ_n is an intertwiner between these two isomorphic representations. This means that for any word $w \in A^n$, (i) $w\kappa_n = 0$ is w is not primitive (*i.e.*, $w = v^d$ for some non-trivial divisor d of n), and (ii) if w is primitive, $wc\kappa_n = \omega w\kappa_n$.

Hence, applying κ_n to some set of representatives of circular classes of primitive words, for example to Lyndon words (words which are lexicographically minimal among their circular shifts), we obtain a basis of $L_n(A)$.

There is a closed formula:

$$\kappa_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \omega^{\mathrm{maj}\,(\sigma)}\sigma \tag{3.28}$$

where $\omega = e^{2\mathrm{i}\pi/n}$ and $\mathrm{maj}\,(\sigma) = \sum_{j\in\mathrm{Des}(\sigma)} j$ is the *major index* of σ.

Now, applying κ_n to the Lyndon words appearing in the expansion of $E_n(A)$ obtained from the Eulerian idempotent gives an expansion of $H_n(A)$ on a basis of $L_n(A)$, which can be further expanded in terms of commutators by means of Dynkin's idempotent if needed.

At this point, we are facing three idempotents, which, admittedly, are given by rather different formulas.

It may therefore come as a surprise that the element

$$\varphi_n(q) = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma)\end{bmatrix}_q} q^{\mathrm{maj}\,(\sigma)-\binom{d(\sigma)+1}{2}}\sigma \tag{3.29}$$

is a Lie idempotent, interpolating between our three examples [46].

Apart from the from the fact that $\varphi_n(1) = \phi_n$, none of these properties is evident. Nevertheless, one can show that $\varphi_n(0) = \theta_n$ and $\varphi_n(\omega) = \kappa_n$.

The explanation of this strange fact starts with the observation that the three idempotents do have something in common: they all belong to the descent algebra.

3.6 Lie idempotents in the descent algebra

It is clear from the definitions that ϕ_n and κ_n are in the descent algebra of $\mathfrak{S}_n$. For θ_n, this is also quite easy to see:

$$2\theta_2 = \begin{array}{ll}\boxed{1}&\boxed{2}\end{array} - \begin{array}{l}\boxed{2}\\\boxed{1}\end{array}$$

$$3\theta_3 = \begin{array}{lll}\boxed{1}&\boxed{2}&\boxed{3}\end{array} - \left(\begin{array}{ll}\boxed{2}&\\\boxed{1}&\boxed{3}\end{array} + \begin{array}{ll}\boxed{3}&\\\boxed{1}&\boxed{2}\end{array}\right) + \begin{array}{l}\boxed{3}\\\boxed{2}\\\boxed{1}\end{array}$$

$$4\theta_4 = \underbrace{\begin{array}{llll}\boxed{1}&\boxed{2}&\boxed{3}&\boxed{4}\end{array}}_{D_4} - \underbrace{\left(\begin{array}{lll}\boxed{2}&&\\\boxed{1}&\boxed{3}&\boxed{4}\end{array} + \begin{array}{lll}\boxed{3}&&\\\boxed{1}&\boxed{2}&\boxed{4}\end{array} + \begin{array}{lll}\boxed{4}&&\\\boxed{1}&\boxed{2}&\boxed{3}\end{array}\right)}_{D_{13}}$$

$$- \underbrace{\left(\begin{array}{ll}\boxed{3}&\\\boxed{2}&\\\boxed{1}&\boxed{4}\end{array} + \begin{array}{ll}\boxed{4}&\\\boxed{2}&\\\boxed{1}&\boxed{3}\end{array} + \begin{array}{ll}\boxed{4}&\\\boxed{3}&\\\boxed{1}&\boxed{2}\end{array}\right)}_{D_{112}} - \underbrace{\begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{2}\\\boxed{1}\end{array}}_{D_{1111}}.$$

That is,

$$n\theta_n = \sum_{k=0}^{n-1} (-1)^k D_{1^k\,n-k}. \tag{3.30}$$

3.7 Lie idempotents as noncommutative symmetric functions

As already mentioned,the first really interesting question about noncommutative symmetric functions is perhaps “what are the noncommutative power sums?”. Indeed, the answer to this question is far from being unique.

If one starts from the classical expression

$$\sigma_t(X) = \sum_{n\geq 0} h_n(X)t^n = \exp\left\{\sum_{k\geq 1} p_k \frac{t^k}{k}\right\}, \tag{3.31}$$

one can choose to define noncommutative power sums Φ_k by the same formula

$$\sigma_t(A) = \sum_{n\geq 0} S_n(A)t^n = \exp\left\{\sum_{k\geq 1} \Phi_k \frac{t^k}{k}\right\}, \tag{3.32}$$

but a noncommutative version of the Newton formulas

$$nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + p_n \tag{3.33}$$

which are derived by taking the logarithmic derivative of (3.31) leads to different noncommutative power-sums Ψ_k inductively defined by

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + \Psi_n. \tag{3.34}$$

A bit of computation reveals then that

$$\Psi_n = R_n - R_{1,n-1} + R_{1,1,n-2} - \cdots = \sum_{k=0}^{n-1}(-1)^k R_{1^k,n-k}, \tag{3.35}$$

which is analogous to the classical expression of p_n as the alternating sum of hook Schur functions. Therefore, in the descent algebra, Ψ_n correponds to Dynkin's element, $n\theta_n$.

The Φ_n can also be expressed on the ribbon basis without much difficulty, and one finds

$$\Phi_n = \sum_{|I|=n} \frac{(-1)^{l(I)-1}}{\binom{n-1}{l(I)-1}} R_I \tag{3.36}$$

so that Φ_n corresponds to $n\phi_n$.

The case of Klyachko's idempotent is even more interesting, but to explain it, we shall need the $(1-q)$-tranform, to be defined below.

3.8 The $(1-q)$-transform

In its classical (commutative) version, the $(1-q)$-transform ϑ_q is the algebra endomorphism of Sym defined on the power sums by $\vartheta_q(p_n) = (1-q^n)p_n$. In λ-ring notation, which is particularly convenient for dealing with such transformations, it reads $f(X) \mapsto f((1-q)X)$. One has to pay attention to the abuse of notation in using the same minus sign for the λ-ring and for scalars, though these operations are quite different. That is, ϑ_{-1} maps p_n to $2p_n$ if n is odd, and to 0 otherwise. Thus, $\vartheta_{-1}(f(X)) = f((1-q)X)_{q=-1}$ is not the same as $f((1+1)X) = f(2X)$.

In [46], a consistent definition of $\vartheta_q(F) = F((1-q)A)$ has been introduced as follows. One first defines the complete symmetric functions $S_n((1-q)A)$ via their generating series [46, Definition 5.1]

$$\sigma_t((1-q)A) := \sum_{n\geq 0} t^n S_n((1-q)A) = \sigma_{-qt}(A)^{-1}\sigma_t(A), \tag{3.37}$$

and then ϑ_q is defined as the ring homomorphism such that

$$\vartheta_q(S_n) = S_n((1-q)A). \tag{3.38}$$

It can then be shown [46, Theorem 4.17] that

$$F((1-q)A) = F(A) * \sigma_1((1-q)A). \tag{3.39}$$

For generic q, ϑ_q is an automorphism, and its inverse is the $1/(1-q)$-transform

$$\sigma_t\left(\frac{A}{1-q}\right) = \overleftarrow{\prod_{n\geq 0}} \sigma_{tq^n}(A). \tag{3.40}$$

Computing the image of S_n, one arrives at

$$K_n(q) = (q)_n S_n\left(\frac{A}{1-q}\right) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I(A). \tag{3.41}$$

Hence, Klyachko's element $n\kappa_n = K(\omega)$ is the specialization of this expression at $q = \omega$. This is puzzling: the commutative image of (3.41) is a Hall-Littlewood function ($\tilde{Q}'_{(1^n)}$, precisely), and the specialization of such functions at roots of unity are known. In this case, one gets the power sum p_n [52, Example 7, page 234].

3.9 Hopf algebras enter the scene

At this point, we can see that the commutative images of our three Lie idempotents are the same: $\frac{1}{n}p_n$. The symmetric functions $\frac{1}{n}p_n$ have two significant properties: (i) they are idempotent for the internal product, and (ii) they are primitive elements for the coproduct of Sym.

One may therefore suspect that our Lie idempotents might be primitive elements of **Sym**. That this is true can be verified directly for each of them, but we have the following much stronger result [46]. Let us say that an element e of an algebra is *quasi-idempotent* if $e^2 = c \cdot e$ for some (possibly 0) scalar c.

Theorem 3.2. *Let* $F = \alpha(\pi)$ *be an element of* $\mathbf{Sym}_n$*, where* $\pi \in \Sigma_n$*.*

(i) *The following assertions are equivalent*:

1. π *is a Lie quasi-idempotent*;
2. F *is a primitive element for* Δ;
3. F *belongs to the Lie algebra* $L(\Psi)$ *generated by the* Ψ_n*.*

(ii) *Moreover,* π *is a Lie idempotent iff* $F - \frac{1}{n}\Psi_n$ *is in the Lie ideal* $[L(\Psi)\,,\, L(\Psi)]$.

Thus, Lie idempotents are essentially the same thing as "noncommutative power sums" (up to a factor n), and we shall from now on identify both notions: a Lie idempotent in $\mathbf{Sym}_n$ is a primitive element whose commutative image is p_n/n.

3.10 A one parameter family of Lie idempotents

Theorem 3.2 suggests a recipe for constructing new examples. Start from a known family, *e.g.*, Dynkin elements, and take its image by a bialgebra automorphism, *e.g.*, ϑ_q^{-1}. The result is then automatically a sequence of Lie idempotents. In the case under consideration, we get

$$\begin{aligned}\varphi_n(q) &= \frac{1-q^n}{n}\Psi_n\left(\frac{A}{1-q}\right) \\ &= \frac{1}{n}\sum_{|I|=n}\frac{(-1)^{\ell(I)-1}}{\begin{bmatrix} n-1 \\ \ell(I)-1\end{bmatrix}_q}\, q^{\operatorname{maj}(I)-\binom{\ell(I)}{2}}\; R_I(A),\end{aligned} \tag{3.42}$$

that is, (3.29). The obtention of the closed form in the r.h.s. requires a fair amount of calculation, but the fact that $\varphi_n(q)$ is a Lie idempotent is automatic. This being granted, it is not difficult to show that

$$\varphi_n(0) = \frac{1}{n}\,\Psi_n\ ,\ \varphi_n(\omega) = \frac{1}{n}K_n(\omega)\ ,\ \varphi_n(1) = \frac{1}{n}\,\Phi_n\ . \tag{3.43}$$

Other Hopf automorphisms, like the noncommutative analogs of the transformation

$$f(X) \longrightarrow f\left(\frac{1-t}{1-q}X\right) \tag{3.44}$$

used in the theory of Macdonald polynomials, lead to other families of Lie idempotents. It is not always possible, however, to obtain such a clean closed form for them.

3.11 The iterated q-bracketing and its diagonalization

There is another one-parameter family of Lie idempotents, for which no closed expression is known, but which is of fundamental importance.

The reproducing kernel of ϑ_q, $S_n((1-q)A)$ is easily seen to be the image under α of the iterated q-bracketing operator

$$S_n((1-q)A) = (1-q)\alpha\left([[\cdots[1,2]_q,3]_q,\ldots,n]_q\right), \tag{3.45}$$

a natural q-analog of Dynkin's idempotent. For generic q, this is not an idempotent at all, but an automorphism.

The most important property of ϑ_q is its diagonalization [46, Theorem 5.14]: there is a unique family of Lie idempotents $\pi_n(q)$ with the property

$$\vartheta_q(\pi_n(q)) = (1-q^n)\pi_n(q). \tag{3.46}$$

Moreover, ϑ_q is semi-simple, and its eigenvalues in $\mathbf{Sym}_n$, the nth homogeneous component of **Sym**, are $p_\lambda(1-q) = \prod_i(1-q^{\lambda_i})$ where λ runs over the partitions of n. The projectors on the corresponding eigenspaces are the maps $F \mapsto F * \pi^I(q)$ [46, Section 3.4].

Here are the first values of $\pi_n(q)$:

$$\pi_1(q) = \Psi_1\,,\quad \pi_2(q) = \frac{\Psi_2}{2}\,,\quad \pi_3(q) = \frac{\Psi_3}{3} + \frac{1}{6}\frac{1-q}{(1+2q)}[\Psi_2,\Psi_1]\,,$$

$$\begin{aligned}
\pi_4(q) = {}& \frac{\Psi_4}{4} + \frac{1}{12}\frac{(1-q)(2q+1)}{(1+q+2q^2)}[\Psi_3,\Psi_1] \\
& + \frac{1}{24}\frac{(1-q)^2}{(1+q+2q^2)}[[\Psi_2,\Psi_1],\Psi_1]\,,\\
\pi_5(q) = {}& \frac{\Psi_5}{5} + \frac{1}{20}\frac{(1-q)(3q^2+2q+1)}{(2q^3+q^2+q+1)}[\Psi_4,\Psi_1] \\
& + \frac{1}{30}\frac{(1-q)(q+2)}{(2q^2+2q+1)}[\Psi_3,\Psi_2] \\
& + \frac{1}{60}\frac{(1-q)^2(4q^3+7q^2+7q+2)}{(2q^2+q+2)(2q^3+q^2+q+1)}[[\Psi_3,\Psi_1],\Psi_1] \\
& - \frac{1}{120}\frac{(1-q)^2(4q^2+9q+7)}{(q^2+3q+1)(2q^2+1+2q)}[[\Psi_1,\Psi_2],\Psi_2] \\
& + \frac{1}{120}\frac{(1-q)^3(2q^5+2q^4+q^3+5q^2+9q+6)}{(2q^3-q^2+q+3)(2q^2+q+2)(2q^3+q^2+q+1)} \\
& \cdot [[[\Psi_2,\Psi_1],\Psi_1],\Psi_1].
\end{aligned}$$

The idempotents $\pi_n(q)$ have interesting specializations. The easiest one is $q = 1$:

$$\pi_n(1) = \frac{\Psi_n}{n} . \tag{3.47}$$

This has the strange consequence that, for *any* Lie idempotent $F_n \in \mathbf{Sym}_n$,

$$\lim_{q\to 1} \frac{F_n((1-q)A)}{1-q^n} = \frac{\Psi_n}{n} . \tag{3.48}$$

Next, we have, for ω a primitive n-th root of unity,

$$\pi_n(\omega) = \kappa_n . \tag{3.49}$$

Again, a curious consequence is that

$$\lim_{q\to\omega} (1-q^n)\, F_n\left(\frac{A}{1-q}\right) = \kappa_n, \tag{3.50}$$

for any Lie idempotent $F_n \in \mathbf{Sym}_n$.

To describe the next specialization, we need to introduce a new family of noncommutative power sums. The *noncommutative power sums* of the *third kind* Z_n are defined by

$$\sigma_t(A) = \exp(Z_1 t)\, \exp\left(\frac{Z_2}{2}t^2\right) \dots \exp\left(\frac{Z_n}{n}t^n\right) \dots \tag{3.51}$$

The Fer-Zassenhauss formula (*cf.* [70]) shows that every Z_n is a Lie element. It is also clear that the commutative image of Z_n is p_n.

The first values of Z_n are

$$\begin{aligned}
Z_1 &= \Psi_1\ ,\ Z_2 = \Psi_2\ ,\ Z_3 = \Psi_3 + \frac{1}{2}[\Psi_2, \Psi_1],\\
Z_4 &= \Psi_4 + \frac{1}{3}[\Psi_3, \Psi_1] + \frac{1}{6}[[\Psi_2, \Psi_1], \Psi_1],\\
Z_5 &= \Psi_5 + \frac{1}{4}[\Psi_4, \Psi_1] + \frac{1}{3}[\Psi_3, \Psi_2] + \frac{1}{12}[[\Psi_3, \Psi_1], \Psi_1]\\
&\quad - \frac{7}{24}[\Psi_2, [\Psi_2, \Psi_1]] + \frac{1}{24}[[[\Psi_2, \Psi_1], \Psi_1], \Psi_1].
\end{aligned}$$

Then, we have [19, 46]

$$\pi_n(0) = \frac{Z_n}{n} . \tag{3.52}$$

3.12 Noncommutative symmetric functions and mould calculus

Recall that a *mould*, in the sense of Ecalle [22], is "a function of a variable number of variables", that is, a family of coefficients $M_{\omega_1,\omega_2,\ldots,\omega_r}$ indexed by finite sequences of elements of some set Ω. We shall see that noncommutative symmetric functions provide us with interesting examples of moulds, Ω being here the set of positive integers, and with good illustrations of the basic notions of mould calculus.

Recall that by definition, **Sym** is a graded free associative algebra, with exactly one generator in each degree. We have seen several sequences of generators of special importance, some being composed of primitive elements, other being sequences of divided powers, so that their generating series is grouplike.

Each pair of such sequences (U_n), (V_n) defines two moulds, whose coefficients express the expansions of the V_n on the U^I, and vice-versa. Ecalle's four fundamental symmetries reflect the four possible combinations of the primitive or grouplike characteristics.

If we denote by $\mathcal{L}$ the (completed) primitive Lie algebra of **Sym** and by $\mathcal{G} = \exp\,\mathcal{L}$ the associated multiplicative group, we have the following table

$\mathcal{L} \to \mathcal{L}$	Alternal
$\mathcal{L} \to \mathcal{G}$	Symmetral
$\mathcal{G} \to \mathcal{L}$	Alternel
$\mathcal{G} \to \mathcal{G}$	Symmetrel

The characterization of alternal moulds in terms of shuffles is equivalent to Ree's theorem (*cf.* [64]): *the orthogonal of the free Lie algebra in the dual of the free associative algebra is spanned by proper shuffles.*

A mould can also be interpreted as a nonlinear operator, mapping the formal series $U = \sum_n U_n$ to the formal series V. The composition of moulds is then the usual composition of the corresponding operators. Since the relationship between two sequences of generators of the same type (divided powers or grouplike) can always be written in the form

$$V_n(A) = U_n(XA) \quad (\text{or } V(t) = U(t) * \sigma_1(XA)), \tag{3.53}$$

where X is a virtual alphabet (commutative and ordered, *i.e.*, a specialization of QSym), the composition of alternal or symmetrel moulds can also be expressed by means of the internal product, as

$$W_n(A) = V_n(XA) = U_n(XYA) = U_n * \sigma_1(XA) * \sigma_1(YA) \tag{3.54}$$

(see [46]).

3.12.1 S_n and Λ_n: symmetrel The simplest example just gives the coefficients of the inverse of a generic series regarded as $\lambda_{-t}(A)$. It is a symmetrel mould:

$$S_n = \sum_{I \vDash n} f_I \Lambda^I, \qquad f_I = (-1)^{n-l(I)}. \tag{3.55}$$

3.12.2 S and Ψ: symmetral/alternel The mould

$$f_I = \frac{1}{i_1(i_1+i_2)\ldots(i_1+\ldots i_r)} \tag{3.56}$$

gives the expression of S_n over Ψ^I:

$$S_n = \sum_{I \vDash n} f_I \Psi^I. \tag{3.57}$$

If we define the series $\psi(t)$ by

$$\psi(t) = \sum_{n \geq 1} t^{n-1} \Psi_n, \tag{3.58}$$

we have $\sigma'(t) = \sigma(t)\,\psi(t)$, and the above mould expresses the solution of the differential equation in terms of iterated integrals

$$\begin{aligned} \sigma(t) &= 1 + \int_0^t dt_1\, \psi(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2\, \psi(t_2)\psi(t_1) \\ &\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, \psi(t_3)\psi(t_2)\psi(t_1) + \cdots \\ &= T \exp\left\{ \int_0^t \psi(s) ds \right\}, \end{aligned} \tag{3.59}$$

or as Dyson's T-exponential.

Similarly, the relation between Λ_n and Ψ^I is obtained by solving

$$\lambda'(t) = -\psi(t)\lambda(t), \tag{3.60}$$

which yields

$$\Lambda_n = \sum_{I \vDash n} (-1)^{l(I)} f_{\bar{I}} \Psi^I, \tag{3.61}$$

where $\bar{I} = (i_r, \ldots, i_2, i_1)$.

Interestingly, such expressions occur in the application of mould calculus to the linearization of non-resonant vector fields (here in the no-so-exciting one dimensional case, but the generalization is easy).

Here is how it goes[7]. Consider the differential equation

$$\partial_t x = u(x) = x + \sum_{n\geq 1} u_n x^{n+1}. \tag{3.62}$$

We are looking for a formal diffeomorphism

$$\phi(x) = x + \sum_{n\geq 1} \phi_n x^{n+1}, \tag{3.63}$$

tangent to the identity, such that

$$y = \phi(x) \quad \text{satisfies} \quad \partial_t y = y. \tag{3.64}$$

That is, we want to conjugate the vector field $u(x)$ to its linear part. In one dimension, the problem is trivial, but the approach by mould calculus sketched below will work in arbitrary dimensions.

Instead of looking for ϕ, one can try to compute the substitution automorphism

$$F:\ A(x) \mapsto A(\phi(x)). \tag{3.65}$$

From Taylor's formula, we see that F can be written as a series of differential operators

$$F = \sum_{n\geq 0} F_n \tag{3.66}$$

where F_0 is the identity map, and F_n shifts degrees by n, that is, $F_n(x^m) = c_{nm}x^{m+n}$.

Obviously, the differential equation (3.62) can be written as

$$\partial_t x = \left(B_0 + \sum_{n\geq 1} B_n \right) x = Bx, \tag{3.67}$$

where $B_0 = x\partial_x$ and $B_n = u_n x^{n+1}\partial_x$.

The trick consists in looking for F in the form

$$F = \mathrm{Id} + \sum_{i_1,\dots,i_r} M_{i_1,\dots,i_r} B_{i_1} \cdots B_{i_r} = \sum_I M_I B^I. \tag{3.68}$$

The B_n being derivations, the condition that F be an automorphism is that the mould M_I be symmetral.

[7] I am indebted to Frédéric Menous for these explanations.

The conjugation equation reads

$$\partial_t y = \phi'(x)\partial_t x = \phi'(x)u(x) = y = \phi(x), \text{ i.e., } BFx = FB_0x. \quad (3.69)$$

One can solve the operator equation

$$BF = FB_0 \quad (3.70)$$

by remarking that

$$[B_0, B_{i_1} \cdots B_{i_r}] = (i_1 + \cdots + i_r)B^I. \quad (3.71)$$

Then, Equation (3.70) can be written

$$[B_0, F] = -CF, \quad C = \sum_{n\geq 1} B_n. \quad (3.72)$$

Introducing a homogeneity variable and setting

$$F(t) = \sum_{n\geq 0} F_n t^n \quad \text{and} \quad C(t) = \sum_{n\geq 1} B_n t^{n-1} \quad (3.73)$$

we can finally recast (3.70) as

$$F'(t) = -C(t)F(t), \quad (3.74)$$

which is exactly (3.60), with $F(t)$ in the role of $\lambda(t)$ and $C(t)$ in the role of $\psi(t)$. So the required mould M_I is given by (3.61), which is indeed symmetral by our previous considerations.

3.12.3 An alternal mould: the Magnus expansion The expansion of Ψ_n in the basis (Φ^K) is given by

$$\Psi_n = \sum_{|K|=n} \left[\sum_{i=1}^{\ell(K)} (-1)^{i-1} \binom{\ell(K)-1}{i-1} k_i \right] \frac{\Phi^K}{\ell(K)!\pi(K)}, \quad (3.75)$$

where $\pi(K) = k_1 \cdots k_r$. Using the symbolic notation

$$\begin{aligned} \{\Phi_{i_1} \cdots \Phi_{i_r}, \ F\} &= \operatorname{ad}\Phi_{i_1} \operatorname{ad}\Phi_{i_2} \cdots \operatorname{ad}\Phi_{i_r}(F) \\ &= [\Phi_{i_1}, [\Phi_{i_2}, [\ldots[\Phi_{i_r}, \ F]\ldots]]] \end{aligned} \quad (3.76)$$

and the classical identity

$$e^a b e^{-a} = \sum_{n\geq 0} \frac{(\operatorname{ad} a)^n}{n!} b = \{e^a, \ b\}, \quad (3.77)$$

we obtain

$$\psi(t) = \sum_{n\geq 0} \frac{(-1)^n}{(n+1)!}\{\Phi(t)^n\,,\ \Phi'(t)\} = \left\{\frac{1-e^{-\Phi(t)}}{\Phi(t)}\,,\ \Phi'(t)\right\} \tag{3.78}$$

which by inversion gives the Magnus formula:

$$\Phi'(t) = \left\{\frac{\Phi(t)}{1-e^{-\Phi(t)}}\,,\ \psi(t)\right\} = \sum_{n\geq 0} \frac{B_n}{n!}(\mathrm{ad}\,\Phi(t))^n\,\psi(t) \tag{3.79}$$

the B_n being the Bernoulli numbers (see [36] and [46] for two diffferent derivations of these identities).

3.12.4 Another alternal mould: the continuous BCH expansion The expansion of $\Phi(t)$ in the basis (Ψ^I) is given by the series

$$\Phi(t) = \sum_{r\geq 1}\int_0^t dt_1 \cdots \int_0^{t_{r-1}} dt_r \sum_{\sigma\in\mathbf{S}_r} \frac{(-1)^{d(\sigma)}}{r}\binom{r-1}{d(\sigma)}^{-1} \psi(t_{\sigma(r)})\cdots\psi(t_{\sigma(1)})\,. \tag{3.80}$$

Thus, the coefficient of $\Psi^I = \Psi_{i_1}\cdots\Psi_{i_r}$ in the expansion of Φ_n is equal to

$$n\int_0^1 dt_1 \cdots \int_0^{t_{r-1}} dt_r \sum_{\sigma\in\mathbf{S}_r} \frac{(-1)^{d(\sigma)}}{r}\binom{r-1}{d(\sigma)}^{-1} t_{\sigma(r)}^{i_1-1}\cdots t_{\sigma(1)}^{i_r-1}\,. \tag{3.81}$$

It is worth observing that this expansion, together with a simple expression of Ψ_n in terms of the dendriform operations of **FQSym**, recently led Ebrahimi-Fard, Manchon, and Patras [21], to an explicit solution of the Bogoliubov recursion for renormalization in Quantum Field Theory.

3.12.5 Moulds related to the Fer-Zassenhaus expansion Recall that the noncommutative power sums of the third kind Z_n are defined by [46]

$$\sigma_t(A) = \exp(Z_1\,t)\,\exp\left(\frac{Z_2}{2}\,t^2\right)\ \ldots\ \exp\left(\frac{Z_n}{n}\,t^n\right)\ \ldots \tag{3.82}$$

This also defines interesting alternal moulds. There is no known expression for Z_n on the Ψ^I, but Goldberg's explicit formula (see [64]) for the Hausdorff series gives the decomposition of Φ_n on the basis Z^I.

3.12.6 A one-parameter family We have seen that

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}} q^{\operatorname{maj}(\sigma) - \binom{d(\sigma)+1}{2}} \sigma \tag{3.83}$$

is a Lie idempotent. Define a nonlinear operator $h(t) \mapsto E_q[h(t)]$, where $h(t) = \sum_{n\geq 1} H_n t^{n-1}$ by

$$E_q[h(t)] = \sum_I c_I(q) H^I t^{|I|}. \tag{3.84}$$

Then,

$$E_1[h(t)] = \exp \int_0^t h(s)ds \tag{3.85}$$

while E_0 is Dyson's chronological exponential

$$\begin{aligned} E_0[h(t)] &= T \exp \int_0^t h(s)ds \\ &= 1 + \int_0^t dt_1 h(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 h(t_2) h(t_1) + \cdots \end{aligned} \tag{3.86}$$

Thus, we interpolate between the chronological exponential and the true exponential.

3.12.7 Another one-parameter family Recall that there is a unique sequence $\pi_n(q)$ of Lie idempotents which are left and right eigenvectors of $\sigma_1((1-q)A)$ for the internal product:

$$\sigma_1((1-q)A) * \pi_n(q) = \pi_n(q) * \sigma_1((1-q)A) = (1-q^n)\pi_n(q) \tag{3.87}$$

and that these elements have the specializations

$$\pi_n(1) = \frac{\Psi_n}{n}, \pi_n(\zeta) = \frac{1}{n} K_n(\zeta),\ \pi_n(0) = \frac{1}{n} Z_n. \tag{3.88}$$

In particular, the associated alternal moulds provide an interpolation between Dyson's T-exponential and the Fer-Zassenhaus expansion.

3.13 Noncommutative symmetric functions and alien calculus

There is also a rather surprising occurence of **Sym** in the theory of resurgent functions [22–24][8].

[8] The content of this section is the result of discussions with Frédéric Menous and David Sauzin.

The algebra RESUR($\mathbb{R}^+//\mathbb{N}$, int.) of resurgent functions on $\mathbb{R}^+$ with singularities at positive integers is defined as the space of functions $\hat{\varphi}$ which (i) are defined and holomorphic on $(0, 1)$, (ii) are analytically continuable along any path that follows $\mathbb{R}^+$ and dodges any point of $\mathbb{N}^*$ to the left or to the right, without ever going back, and (iii) all determinations are locally integrable on $\mathbb{R}^+$.

It can be shown that this is a convolution algebra.

A sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \{\pm\}^{n-1}$ encodes a path like this one

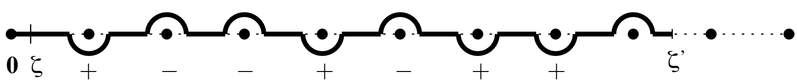


Figure 3.1.

and defines an operator $D_{\varepsilon\bullet}$ on RESUR($\mathbb{R}^+//\mathbb{N}$, int.) by

$$D_{\varepsilon\bullet}\hat{\varphi} = \hat{\varphi}^{\varepsilon+}(\zeta + l(\varepsilon\bullet)) - \hat{\varphi}^{\varepsilon-}(\zeta + l(\varepsilon\bullet)). \tag{3.89}$$

Here, $l(\varepsilon\bullet) = n$ is the length of the sequence $(\varepsilon_1, \ldots, \varepsilon_{n-1}, \bullet)$, $\varepsilon+ = (\varepsilon_1, \ldots, \varepsilon_{n-1}, +1)$ and $\varepsilon- = (\varepsilon_1, \ldots, \varepsilon_{n-1}, -1)$.

The composition of such operators is given by

$$D_{\mathbf{a}\bullet} D_{\mathbf{b}\bullet} = D_{\mathbf{b}+\mathbf{a}\bullet} - D_{\mathbf{b}-\mathbf{a}\bullet} \tag{3.90}$$

which is, up to a sign, the product formula for noncommutative ribbon Schur functions

$$R_I \cdot R_J = R_{I \cdot J} + R_{I \triangleright J} \tag{3.91}$$

where $I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s)$ and $I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$ (note that this formula is fairly obvious from the interpretation in terms of words).

The signs can be taken into account, and if we denote by ALIEN the algebra spanned by the $D_{\varepsilon\bullet}$, we have a natural isomorphism of algebras

$$\text{ALIEN} \longrightarrow \mathbf{Sym} \tag{3.92}$$

which in turn is also an isomophism of Hopf algebras, for the Hopf structure introduced by Ecalle on ALIEN. It is given by

$$D_{\varepsilon\bullet} \quad \leftrightarrow \quad \varepsilon_1 \ldots \varepsilon_{n-1} R_\varepsilon \tag{3.93}$$

where the ribbon Schur function R_ε is obtained by reading backwards the sequence $\varepsilon+$:

$$\varepsilon+ \;=\; +--+++-+ \quad \to \quad R_\varepsilon = \begin{array}{|c|c|c|c|c|} \cline{1-1} + & \multicolumn{4}{c}{} \\ \hline - & + & + & + & \multicolumn{1}{c}{} \\ \cline{1-4} \multicolumn{3}{c|}{} & - & \multicolumn{1}{c}{} \\ \cline{4-5} \multicolumn{3}{c|}{} & - & + \\ \cline{4-5} \end{array} \tag{3.94}$$

Under this isomorphism[9],

$$\begin{aligned} \Delta_n^+ &= D_{+\ldots+\bullet} \leftrightarrow S_n \\ \Delta_n^- &= -D_{-\ldots-\bullet} \leftrightarrow (-1)^n \Lambda_n \\ \Delta_n &= \sum_{\varepsilon\in\mathcal{E}_{n-1}} \frac{p!q!}{(p+q+1)!} D_{\varepsilon\bullet} \leftrightarrow \frac{1}{n}\Phi_n. \end{aligned} \tag{3.95}$$

Given these identifications, it is not so surprising that ALIEN can be given Hopf algebra structure, for which Δ^+ and Δ^- are grouplike, and Δ primitive. However, the analytical definition of the coproduct is not trivial [22]. Grouplike elements are the alien automorphisms, and primitives are the alien derivations.

Thus, alien derivations correspond, via the isomorphism with **Sym**, to Lie idempotents in descent algebras. Nontrivial examples are known on both sides. For example, alien derivations from the Catalan family [25]:

$$\mathrm{Dam}_n = \sum_{l(\varepsilon\bullet)=n} \mathrm{ca}^\varepsilon D_{\varepsilon\bullet}, \tag{3.96}$$

where

$$\mathrm{ca}_n = \frac{(2n)!}{n!(n+1)!}, \tag{3.97}$$

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) = (\pm)^{n_1}(\mp)^{n_2}(\pm)^{n_3}\ldots(\varepsilon_n)^{n_s} \quad (n_1+\ldots+n_s=n), \tag{3.98}$$

$$\mathrm{ca}^\varepsilon = \mathrm{ca}_{n_1}\mathrm{ca}_{n_2}\ldots\mathrm{ca}_{n_s}. \tag{3.99}$$

For example,

$$\mathrm{Dam}_4 = 5R_4 - 5R_{1111} - 2R_{13} + 2R_{211} - 2R_{31} + 2R_{112} - R_{22} + R_{121}.$$

The corresponding Lie idempotents were not known, and up to now, no natural way to prove their primitivity in **Sym** is known either.

On another hand, one may ask whether there is any application in alien calculus of the q-Solomon idempotent or of the other non trivial examples already presented.

4 Permutations and free quasi-symmetric functions

4.1 Free quasi-symmetric functions

To go further, we need larger algebras. The simplest one is based on permutations. It is large enough to contain algebras based on binary trees or

[9] The symbols Δ here have nothing to do with the coproduct!

on Young tableaux. To accomodate other kinds of trees, one can imitate its construction, starting from special words generalizing permutations.

The algebra of permutations has been first investigated by Reutenauer [64], as a the convolution algebra of graded $GL(V)$-endomorphisms of a tensor algebra $T(V)$. Its Hopf algebra structure has been explicited in [54].

There is however a more direct and elementary approach [16]. Recall that our noncommutative ribbon Schur function R_I has two interpretations:

(i) as the sum of words of shape I in the free associative algebra, and
(ii) as the sum of permutations of shape I in the group algebra of the symmetric group.

So, one may ask whether is is possible to associate with each word of shape I a permutation of shape I, so as to reconcile both approaches, and interpret each permutation as a sum of words.

This is indeed possible, and the solution is given by the classical *standardization process*, familiar in combinatorics and in computer science. The *standardized word* $\mathrm{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \mathrm{Std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word. For example, $\mathrm{Std}(bbacab) = 341625$.

Obviously, $\mathrm{std}(w)$ has the same descents as w. We can now define polynomials

$$\mathbf{G}_\sigma(A) := \sum_{\mathrm{std}(w)=\sigma} w. \tag{4.1}$$

It is not hard to check that the linear span of these polynomials is a subalgebra, denoted by $\mathbf{FQSym}(A)$, an acronym for *Free Quasi-Symmetric functions*.

Since the definition of the $\mathbf{G}_\sigma(A)$ involves only a totally ordered alphabet A, we can apply it to an ordinal sum $A + B$, and as in the case of **Sym**, this defines a coproduct if we assume that A commutes with B. Clearly, this coproduct is coassociative and multiplicative, so that we have a graded (and connected) bialgebra, hence again a Hopf algebra. By definition,

$$R_I(A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(A) \tag{4.2}$$

so that **Sym** embeds into **FQSym** as a Hopf subalgebra.

It is also easy to check that **FQSym** is self-dual. If we set $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$ and $\langle \mathbf{F}_\sigma, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau}$, then $\langle FG, H\rangle = \langle F \otimes G, \Delta H\rangle$.

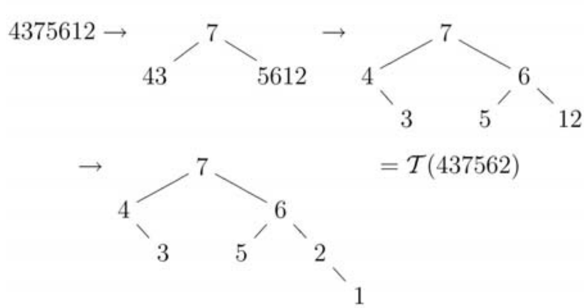
The graded dual of **Sym** is the (commutative) algebra QSym of *quasi-symmetric functions* (*cf.* [36,37,54]). Hence, we have a surjective homomorphism $\mathbf{FQSym}^* \to \mathrm{QSym}$. Its description is particularly simple: it consists in replacing our noncommuting variables a_i by commuting ones x_i. Then, $\mathbf{F}_\sigma(X)$ depends only on the descent composition $I = C(\sigma)$, and this can be taken as a definition of the quasi-symmetric function F_I [32].

One can embed many combinatorial Hopf algebras into **FQSym**, by classifying permutations according to various features (in the case of **Sym**, the feature was the descent set). For example, one can build a Hopf algebra of standard Young tableaux by means of the Robinson-Schensted correspondence [61]. When interpreted in **FQSym**, this construction proves in one stroke the famous Littlewood-Richardson rule for the multiplication of Schur functions [16]. Replacing standard tableaux by binary trees and the Robinson-Schensted insertion by the binary search tree insertion, familiar to computer scientists, one arrives [38] at the Loday-Ronco Hopf algebra of binary trees [49].

4.2 From permutations to binary trees

The original construction of Loday and Ronco, motivated by operadic considerations, was based on the notion of *decreasing tree* of a permutation.

The decreasing tree $\mathcal{T}(\sigma)$ of a permutation (of some set of integers, not necessarily $\{1, \ldots, n\}$), is defined as follows. If $\sigma = (i)$, the tree consists of a unique vertex labelled i. Otherwise, if n is the greatest letter of σ, write as a word $\sigma = \alpha \cdot n \cdot \upsilon$. Then $\mathcal{T}(\sigma)$ has a root labelled n, $\mathcal{T}(\alpha)$ as left subtree and $\mathcal{T}(\beta)$ as right subtree. For example,



Now, define

$$\mathbf{P}_T = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma \qquad (4.3)$$

and

$$\mathbf{PBT} = \bigoplus \mathbb{K}\mathbf{P}_T. \tag{4.4}$$

As the reader may have guessed, **PBT** is a Hopf subalgebra of **FQSym**. In the operadic language, it is the free dendriform algebra on one generator, and it is precisely for this reason that Loday and Ronco constructed it.

But other questions can lead to the same algebra. We have already mentioned the combinatorics of binary search trees, we shall now see how it arises from a "formal Dyson-Schwinger equation". Other combinatorial Hopf algebras can be obtained by similar considerations, see in particular [27], where continuous families interpolating between symmetric functions and the Faà di Bruno Hopf algebra (and their noncommutative versions) are found.

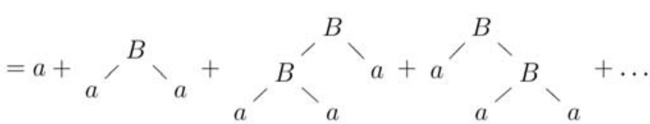
4.3 Trees from functional equations

Consider the functional equation

$$x = a + B(x, x) \tag{4.5}$$

where x lives in some graded associative algebra, and B is a bilinear map such that $\deg B(x, y) > \deg(x) + \deg(y)$. It can be (formally) solved by iterated substitution

$$x = a + B(a, a) + B(B(a, a), a) + B(a, B(a, a)) + \cdots$$



so that its unique solution is

$$x = \sum_{T:\ \text{Complete Binary Tree}} B_T(a) \tag{4.6}$$

For example, $x(t) = \frac{1}{1-t}$ is the unique solution of

$$\frac{dx}{dt} = x^2\,, \quad x(0) = 1 \tag{4.7}$$

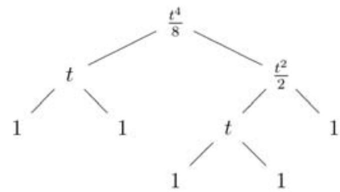
This is equivalent to the fixed point problem

$$x = 1 + \int_0^t x^2(s)ds = 1 + B(x, x) \tag{4.8}$$

where

$$B(x, y) := \int_0^t x(s)y(s)ds \tag{4.9}$$

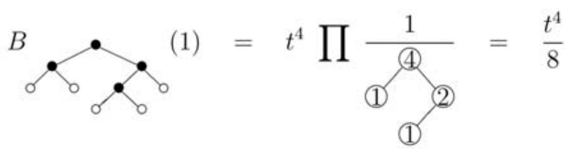
The terms in the tree expansion look like



which yields $B_T(1) = \frac{t^4}{8}$. The general expression is:

$$B_T(1) = t^{\#(T')} \prod_{\bullet \in T'} \frac{1}{HL(\bullet)} \tag{4.10}$$

where T' denotes the incomplete tree obtained by removing the leaves of T, and $HL(\bullet)$ is the size of the subtree rooted at $\bullet$. For example,



(4.11)

Now, it is known that the number of permutations whose decreasing tree has shape T is $n!B_T(1)$ [44]. In **FQSym**, we have

$$\mathbf{G}_1^n = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{G}_\sigma \tag{4.12}$$

and

$$\phi : \mathbf{G}_\sigma \longmapsto \frac{t^n}{n!} \quad \text{for } \sigma \in \mathfrak{S}_n \tag{4.13}$$

is a homomorphism of algebras. Hence, on can write

$$x(t) = \frac{1}{1-t} = \phi\left((1 - \mathbf{G}_1)^{-1}\right). \tag{4.14}$$

Actually, one can find a derivation ∂ of **FQSym** such that $\mathbf{X} = (1 - \mathbf{G}_1)^{-1}$ satisfies $\partial \mathbf{X} = \mathbf{X}^2$. Moreover, there is a bilinear map B such that $\partial B(f, g) = fg$. Hence, $\mathbf{X}$ is the unique solution of $\mathbf{X} = 1 + B(\mathbf{X}, \mathbf{X})$.

For this equation, the tree expansion produces

$$B_T(1) = \mathbf{P}_T, \tag{4.15}$$

the Loday-Ronco basis. This approach may motivate the introduction of $\mathbf{P}_T$. Moreover, it can lead to new combinatorial results by using more sophisticated homomorphisms. In particular, one can recover the Björner-Wachs q-analogues of Knuth's formula from $x = 1 + B_q(x, x)$, with

$$B_q(x, y) = \int_0^t x(s) \cdot y(qs)\, d_q s \tag{4.16}$$

(the Jackson q-integral) [40], and even obtain (q, t)-analogues [59].

5 Parking functions and other algebras

5.1 Special words and normalization algorithms

A whole class of combinatorial Hopf algebras whose operations are usually described in terms of some elaborate surgery on combinatorial objets are in fact just subalgebras of $\mathbb{K}\langle A\rangle$:

- **Sym**: $R_I(A)$ is the sum of all words with the same *descent set*.
- **FQSym**: $\mathbf{G}_\sigma(A)$ is the sum of all words with the same *standardization*.
- **PBT**: $\mathbf{P}_T(A)$ is the sum of all words with the same *binary search tree*.

To these examples, one can add:

- **WQSym**: $\mathbf{M}_u(A)$ is the sum of all words with the same *packing*.
- It contains the free tridendriform algebra one one generator, based on sums of words with the same *plane tree*.
- **PQSym**: based on parking functions (sum of all words with the same *parkization*).

In all cases, the product is the ordinary product of polynomials, and the coproduct is $A + B$. Basic information on **WQSym** can be found in [40]. We shall conclude with a brief presentation of **PQSym**.

5.2 A Hopf algebra on parking functions

A parking function of length n is a word over w over $[1, n]$ such that in the *sorted word* $w^\uparrow$, the ith letter is $\leq i$.

For example $w = 52321$ is a parking function since $w^\uparrow = 12235$, but 52521 is not. Again, this notion comes from computer science.

To replace standardization, one can define a parkization algorithm: sort w, shift the smallest letter if it is not 1, then if necessary, shift the second smallest letter of a minimal amount, and so on. Then put each letter back in its original place.

For example, with $w = (5, 7, 3, 3, 13, 1, 10, 10, 4)$, we have $w^{\uparrow} = (1, 3, 3, 4, 5, 7, 10, 10, 13)$, $p(w)^{\uparrow} = (1, \mathbf{2}, \mathbf{2}, 4, 5, \mathbf{6}, \mathbf{7}, \mathbf{7}, \mathbf{9})$, and finally $p(w) = (4, 6, 2, 2, 9, 1, 7, 7, 3)$.

The sums of words having the same parkization

$$\mathbf{G_a} = \sum_{p(w)=\mathbf{a}} w \tag{5.1}$$

span a subalgebra of the free associative algebra, which can once more be turned into a Hopf algebra by the $A + B$ trick [57]. This is **PQSym**, the Hopf algebra of Parking Quasi-Symmetric functions. It has many interesting quotients and subalgebras.

The number of parking functions of length n is $\mathrm{PF}_n = (n + 1)^{n-1}$. Bijections with trees are known, and parking functions are related to the combinatorics of Lagrange inversion. They also explain the noncommutative Lagrange inversion, which can be interpreted as the antipode of the Hopf algebra of noncommutative formal diffeomorphisms (noncommutative version of the Faà di Bruno algebra [10]).

5.3 A Catalan algebra related to quasi-symmetric functions

As our last example, we shall see a cocommutative Hopf algebra based on a Catalan set, whose dual refines quasi-symmetric functions.

It is somewhat natural to group the parking functions $\mathbf{a}$ according the the sorted word $\pi = \mathbf{a}^{\uparrow}$, which occurs in their definition, and also in the noncommutative Lagrange inversion formula. Then, the sums

$$\mathbf{P}^{\pi} = \sum_{\mathbf{a}^{\uparrow}=\pi} \mathbf{G_a} \tag{5.2}$$

span a Hopf subalgebra **CQSym** of **PQSym**. We have dim$\mathbf{CQSym}_n = c_n$ (Catalan numbers 1,1,2,5,14). Moreover,$\mathbf{P}^{\pi}$ is a multiplicative basis: $\mathbf{P}^{11}\mathbf{P}^{1233} = \mathbf{P}^{113455}$ (shifted concatenation). It is free over a Catalan set $\{1, 11, 111, 112, \dots\}$ (nondecreasing parking functions with an extra 1 on the left). It is cocommutative. So, by the Cartier-Milnor-Moore theorem, it must be isomorphic to the Grossman-Larson algebra of ordered trees [34]. However, it has a very different definition (no trees at all!). This definition reveals an interesting property of its (commutative) dual: **CQSym*** contains QSym in a natural way

Recall the monomial symmetric functions $m_\lambda = \Sigma x^\lambda$. They can be cut into pieces

$$m_\lambda = \sum_{I^{\downarrow}=\lambda} M_I, \quad M_I(X) = \sum_{j_1<j_2<\dots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r} \tag{5.3}$$

The M_I are the monomial quasi-symmetric functions [37].

Let $\mathcal{M}_\pi$ be the dual basis of $\mathbf{P}^\pi$. It can be realized by polynomials:

$$\mathcal{M}_\pi = \sum_{p(w)=\pi} \underline{w} \tag{5.4}$$

where $\underline{w}$ means commutative image ($a_i \to x_i$). For example,

$$\begin{aligned}
\mathcal{M}_{111} &= \sum_i x_i^3 \\
\mathcal{M}_{112} &= \sum_i x_i^2 x_{i+1} \\
\mathcal{M}_{113} &= \sum_{i,j;\, j\geq i+2} x_i^2 x_j \\
\mathcal{M}_{122} &= \sum_{i,j;\, i<j} x_i x_j^2 \\
\mathcal{M}_{123} &= \sum_{i,j,k;\, i<j<k} x_i x_j x_k .
\end{aligned}$$

Then,

$$M_I = \sum_{t(\pi)=I} \mathcal{M}_\pi , \tag{5.5}$$

where $t(\pi)$ is the composition obtained by counting the occurences of the different letters of π. For example,

$$M_3 = \mathcal{M}_{111}, \quad M_{21} = \mathcal{M}_{112} + \mathcal{M}_{113}, \quad M_{12} = \mathcal{M}_{122}.$$

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